CENTERS OF UNIVERSAL ENVELOPING ALGEBRAS OF LIE SUPERALGEBRAS IN PRIME CHARACTERISTIC

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Abstract. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be a basic classical Lie superalgebra over an algebraically closed field k of characteristic $p > 2$, and G be an algebraic supergroup satisfying $\mathrm{Lie}(G) = \mathfrak{g}$, with the purely even subgroup G_{ev} which is a reductive group. In this paper, we prove that the center $\mathcal{Z} := \mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ is a domain, and $U(\mathfrak{g}) \otimes_{\mathcal{Z}} \mathrm{Frac}(\mathcal{Z})$ is a simple superalgebra over $\mathrm{Frac}(\mathcal{Z})$. And then we prove that the quotient field of $\mathcal{Z}$ coincides with that of the subalgebra generated by the G_{ev}-invariant ring $\mathcal{Z}^{G_{\mathrm{ev}}}$ of $\mathcal{Z}$ and the p-center $\mathcal{Z}_0$ of $U(\mathfrak{g}_{\bar{0}})$. In the case when $\mathfrak{g} = \mathfrak{osp}(1|2n)$, $\mathcal{Z}$ is just generated by $\mathcal{Z}^{G_{\mathrm{ev}}}$ and $\mathcal{Z}_0$. Furthermore, we will demonstrate the precise relation between the smooth points of the maximal spectrum $\mathrm{Maxspec}(\mathcal{Z})$ and the corresponding irreducible modules for $\mathfrak{osp}(1|2)$.

1. Introduction

The center of the universal enveloping algebra of a Lie algebra is essential to the study of the representations and has been known in great depth over an algebraically closed field k of positive characteristic p (cf. [6], [23], [36] and [40], *etc.*). To our best knowledge, there has been little understanding of the center of the universal enveloping superalgebra of a Lie superalgebra over k. Generally speaking, the structure for the latter seems to be much more complicated. There are lots of zero-divisors in the universal enveloping algebras in the supercase. And the centers of those universal enveloping algebras are by definition supercommutative, therefore, generally containing lots of odd elements. The present paper is concerned with this issue. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be a basic classical Lie superalgebra over k. There is an algebraic supergroup G satisfying $\mathrm{Lie}(G) = \mathfrak{g}$, with the purely even subgroup G_{ev} which is a reductive group (see §2.3). The center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ turns out to be commutative instead of being supercommutative. Furthermore, this center $\mathcal{Z}(\mathfrak{g})$ is proved not to contain any divisor with desirable properties as below.

Theorem 1.1. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra over k, and $\mathcal{Z} := \mathcal{Z}(\mathfrak{g})$ the center of the universal enveloping algebra $U(\mathfrak{g})$. The following statements hold:*

(1) *$\mathcal{Z}$ is a domain.*

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(2) *The fraction algebra $\mathcal{D}(\mathfrak{g})$ of $U(\mathfrak{g})$ over the fractional field Frac($\mathcal{Z}$) of $\mathcal{Z}$ is simple, as an ordinary associative algebra, therefore a simple superalgebra over Frac($\mathcal{Z}$).*

Thus, we can exploit the arguments in [23] for the ordinary Lie algebra case (resp. for the case of quantum groups at unity root [13]) to the supercase. We will finally obtain the following main result.

Theorem 1.2. *Let $\mathcal{Z}_1$ be the subalgebra of $\mathcal{Z}$ generated by $\mathcal{Z}_0$ and $\mathcal{Z}^{G_{ev}}$, and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and W be the Weyl group of $\mathfrak{g}_{\bar{0}}$. Then*

(1) *Frac($\mathcal{Z}^{G_{ev}}$) is isomorphic to Frac($U(\mathfrak{h})^W$) as fileds.*
(2) *Frac($\mathcal{Z}$) = Frac($\mathcal{Z}_1$).*

For better understanding the above, we recall the geometric features of the center of the universal enveloping algebra of a reductive Lie algebra in prime characteristic. Denote by $\mathrm{Maxspec}(R)$ the maximal spectrum of R for a finitely-generated integral domain R over k. Then, for a reductive Lie algebra $\mathfrak{g}$, the Zassenhaus variety $\mathsf{X} = \mathrm{Maxspec}(\mathcal{Z}(\mathfrak{g}))$ is a normal variety, the smooth points in the variety corresponds to the irreducible representations of maximal dimension. Such an algebraic-geometric feature connects the key information of representations of $\mathfrak{g}$ to the geometry of X (cf. [6, 23, 25, 36]). But in the Lie superalgebra case, the situation seems to be much complicated because of appearance of zero-divisor, and because of the absence of normality of the Zassenhaus variety. It's a great challenge to make some clear and general investigation on the connection between modular representations of classical Lie superalgebras and the geometry of the corresponding Zassenhaus varieties. The study of the former was initiated by Wang and Zhao (ref. [38] and [39]).

Nevertheless, we can provide more information in the special case when $\mathfrak{g} = \mathfrak{osp}(1|2n)$. In such a case, Theorem 1.2 has a stronger version that $\mathcal{Z}^{G_{\mathrm{ev}}}$ is isomorphic to $U(\mathfrak{h})^W$, and $\mathcal{Z}$ coincides with $\mathcal{Z}_1$ (see Theorem 6.5). Furthermore, for $\mathfrak{g} = \mathfrak{osp}(1|2)$ we can describe the smooth locus of $\mathcal{Z}$ to be the union of B and $\mathrm{Ann}_{\mathcal{Z}}(L(\frac{p-1}{2}))$ where $\mathsf{B} = \{\mathrm{Ann}_{\mathcal{Z}}(M) \mid M \text{ is a simple } U_\chi(\mathfrak{g})\text{-module with } \chi \in \mathfrak{g}_{\bar{0}}{}^* \text{ regular}\}$ (see Theorem 7.15).

It's necessary to remind the readers of comparing the center structure of enveloping algebras of Lie superalgebras and Harish-Chandra homomorphism in the modular case (as in the present paper) with that in the complex number case. For the latter, one can be refereed to [12] and [22], *etc.*.

2. Preliminaries

We always assume $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is a finite-dimensional Lie superalgebra over an algebraically closed field k of characteristic $p > 2$. Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$.

By vector spaces, subalgebras, ideals, modules, and submodules etc. we mean in the super sense unless otherwise specified, throughout the paper.

2.1. **Restricted Lie superalgebras.** A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is called a restricted Lie superalgebra if $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a restricted module for $\mathfrak{g}_{\bar{0}}$. Momentarily, we fix a basis $\{x_1, \cdots, x_s\}$ of $\mathfrak{g}_{\bar{0}}$ and a basis $\{y_1, \cdots, y_t\}$ of $\mathfrak{g}_{\bar{1}}$ for the arguments below.

Denote by $\mathcal{Z}(\mathfrak{g})$ the center of $U(\mathfrak{g})$, i.e. $\mathcal{Z}(\mathfrak{g}) := \{u \in U(\mathfrak{g}) \mid \mathrm{ad}x(u) = 0 \ \ \forall x \in \mathfrak{g}\}$. It's easily seen that for any $z \in \mathcal{Z}(\mathfrak{g})$, $z = z_0 + z_1, z_i \in U(\mathfrak{g})_{\bar{i}}, i = 0, 1$, then $z_i \in \mathcal{Z}(\mathfrak{g})$. This is to say, $\mathcal{Z}(\mathfrak{g})$ is a $\mathbb{Z}_2$-graded subalgebra of $U(\mathfrak{g})$. In the sequel, we will often write $\mathcal{Z}(\mathfrak{g})$ simply as $\mathcal{Z}$ provided that the context is clear.

By the definition of a restricted Lie superalgebra, the whole p-center of $U(\mathfrak{g}_0)$ falls in $\mathcal{Z}$, this is to say, $x^p - x^{[p]} \in \mathcal{Z}, \forall x \in \mathfrak{g}_{\bar{0}}$. We denote the p-center by $\mathcal{Z}_0$. Set $\xi_i = x_i^p - x_i^{[p]}, i = 1, \ldots, s$. The p-center $\mathcal{Z}_0$ is a polynomial ring $k[\xi_1, \ldots, \xi_s]$ generated by $\xi_1, \ldots, \xi_s$.

By the PBW theorem, one easily knows that the enveloping superalgebra $U(\mathfrak{g})$ is a free module over $\mathcal{Z}_0$ with basis

$$x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t}, 0 \leq a_i \leq p-1,\ b_j \in \{0,1\} \text{ for } i = 1, \cdots, s, j = 1, \cdots, t.$$

2.2. **Reduced enveloping algebras of restricted Lie superalgebras.** Let V be a simple $U(\mathfrak{g})$-module for a restricted Lie algebra $\mathfrak{g}$ as the previous subsection. Then Schur's Lemma implies that for each $x \in \mathfrak{g}_{\bar{0}}$, $x^p - x^{[p]}$ acts by a scalar $\chi(x)^p$ for some $\chi \in \mathfrak{g}_{\bar{0}}{}^*$. We call such a χ the p-character of V. For a given $\chi \in \mathfrak{g}_{\bar{0}}{}^*$, let I_χ be the ideal of $U(\mathfrak{g})$ generated by the even central elements $x^p - x^{[p]} - \chi(x)^p$. Generally, a module is called a χ-reduced module for a given $\chi \in \mathfrak{g}^*$ if for any $x \in \mathfrak{g}_{\bar{0}}$, $x^p - x^{[p]}$ acts by a scalar $\chi(x)^p$. All χ-reduced modules for a given $\chi \in \mathfrak{g}^*$ constitute a full subcategory of the $U(\mathfrak{g})$-module category. The quotient algebra $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$ is called the reduced enveloping superalgebra of p-character χ. Obviously, the χ-reduced module category of $\mathfrak{g}$ coincides with $U_\chi(\mathfrak{g})$-module category. The superalgebra $U_\chi(\mathfrak{g})$ has a basis

$$x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t}, 0 \leq a_i \leq p-1; b_j \in \{0,1\} \text{ for } i = 1, \cdots s; j = 1, \cdots, t.$$

In particular, $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}_{\bar{0}}} 2^{\dim \mathfrak{g}_{\bar{1}}}$.

2.3. **Basic classical Lie superalgebras and the corresponding algebraic supergroups.** Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be a basic classical Lie superalgebra over k. We list all the basic classical Lie superalgebras and their even parts over k with the restriction on p (cf. [21], [38]).

Basic classical Lie superalgebra $\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	characteristic of k
$\mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$	$p > 2$
$\mathfrak{sl}(m\|n)$	$\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus k$	$p > 2, p \nmid (m-n)$
$\mathfrak{osp}(m\|n)$	$\mathfrak{so}(m) \oplus \mathfrak{sp}(n)$	$p > 2$
F(4)	$\mathfrak{sl}(2) \oplus \mathfrak{so}(7)$	$p > 15$
G(3)	$\mathfrak{sl}(2) \oplus \mathrm{G}_2$	$p > 15$
$\mathrm{D}(2,1,\alpha)$	$\mathfrak{sl}(2) \oplus \mathfrak{sl} \oplus \mathfrak{sl}(2)$	$p > 3$

There is an algebraic supergroup G satisfying $\mathrm{Lie}(G) = \mathfrak{g}$ such that

(1) G has an subgroup scheme G_{ev} which is an ordinary connected reductive group with $\mathrm{Lie}(G_{\mathrm{ev}}) = \mathfrak{g}_{\bar{0}}$;
(2) There is well-defined action of G_{ev} on $\mathfrak{g}$, reducing to the adjoint action of $\mathfrak{g}_{\bar{0}}$.

The above algebraic supergroup can be constructed as a Chevalley supergroup in [14]. The pair $(G_{\mathrm{ev}}, \mathfrak{g})$ constructed in this way is called a Chevalley super Harish-Chandra pair (cf. [15, 3.13] and [14, 5.5.6]). Partial results on G and G_{ev} can be refereed to [5, Ch. II.2], [10], [14], [15, §3.3], and [35, Ch.7], *etc.*. In the present paper, we will call G_{ev} the purely even subgroup of G. One easily knows that $\mathfrak{g}$ is a restricted Lie superalgebra (cf. [30, Lemma 2.2] and [31]).

Recall that $\mathrm{Maxspec}(\mathcal{Z}_0)$ is G_{ev}-equivalently isomorphic to $\mathfrak{g}_{\bar{0}}^{*(1)}$ (cf. [23, Lemma 4.1]), where the superscript $^{(1)}$ means the Frobenius twist (cf. [23, §4]). In the sequel, we will identify $\mathrm{Maxspec}(\mathcal{Z}_0)$ with $\mathfrak{g}_{\bar{0}}^{*(1)}$. Therefore, we can consider the coadjoint action of G_{ev} on $\mathrm{Maxspec}(\mathcal{Z}_0)$. And, the adjoint action of G_{ev} on $\mathfrak{g}$ can be extended to the universal enveloping superalgebra $U(\mathfrak{g})$, as automorphisms.

2.4. **Root space decomposition.** Under the restrictions on p as above, we have the following proposition for all basic classical Lie superalgebras:

Proposition 2.1. *(cf.* [21, 2.5.3] *and* [38, §2]*) Let $\mathfrak{g}$ be a basic classical Lie superalgebra over k. Then $\mathfrak{g}$ has its root space decomposition with respect to a Cartan subalgebra $\mathfrak{h}$: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus\limits_{\alpha\in\Delta} \mathfrak{g}_\alpha$, which satisfies the following properties:*

(1) *Each root space $\mathfrak{g}_\alpha$ is one-dimensional.*
(2) *For $\alpha, \beta \in \Delta$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha + \beta \in \Delta$.*
(3) *There is a non-degenerate super-symmetric invariant bilinear form $(\ , \ )$ on $\mathfrak{g}$. And $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha \neq -\beta$; the form $(\ , \ )$ determines a non-degenerate pairing of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ and the restriction of $(\ , \ )$ on $\mathfrak{h}$ is non-degenerate.*
(4) *$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, where h_α is a non-zero vector determined by $(h_\alpha, h) = \alpha(h), \forall\ h \in \mathfrak{h}$.*

3. The center is a domain: the proof of Theorem 1.1

Throughout the section, we will maintain the notations and assumptions as in Theorem 1.1. In particular, $\mathfrak{g}$ will always be assumed to be a basic classical Lie superalgebra over k, as listed in §2.

3.1. **All central elements in $U(\mathfrak{g})$ are even.** We first need the notion of anti-center $\mathcal{A}(L)$ of $U(L)$ for a Lie superalgebra L introduced in [16]. There is a twist adjoint action ad_{t} of L on $U(L)$ which is defined via $\mathrm{ad}_{\mathsf{t}}x(u) = xu - (-1)^{\bar{x}(\bar{u}+1)}ux$ for homogeneous elements $x \in L_{\bar{x}}$ and $u \in U(L)_{\bar{u}}$ where $\bar{x}, \bar{u} \in \mathbb{Z}_2$. The anticenter $\mathcal{A}(L)$ is defined to be the set of elements of $U(L)$ which is invariant with respect to ad_{t}.

For $\mathcal{A}(L)$, there is a basic fact as below (cf. [16]).

Lemma 3.1. *If $\dim L_{\bar{1}}$ is even* (*odd*)*, then the degree of each element in $\mathcal{A}(L)$ is even* (*odd*)*.*

As a direct application of the above lemma, we have the following observation for a basic classical Lie superalgebra.

Proposition 3.2. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then the following statements hold.*

(1) *Each element in $\mathcal{Z}(\mathfrak{g})$ has even degree, that is to say, $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})_{\bar{0}}$.*[1]
(2) *$\mathcal{Z}(\mathfrak{g})$ is a finitely generated $\mathcal{Z}_0$-module, integral over $\mathcal{Z}_0$. In particular, $\mathcal{Z}(\mathfrak{g})$ is a finitely generated commutative algebras over k.*

Proof. (1) Recall that for a basic classical Lie algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, $\dim \mathfrak{g}_{\bar{1}}$ is even. Lemma 3.1 implies that $\mathcal{A}(\mathfrak{g}) \subset U(\mathfrak{g})_{\bar{0}}$. On the other hand, for each homogeneous elements $z \in \mathcal{Z}(\mathfrak{g})$ and $a \in \mathcal{A}(\mathfrak{g})$, and for homogeneous element $x \in \mathfrak{g}$, one has $x(za) = (-1)^{\bar{z}\bar{x}} zxa = (-1)^{\bar{z}\bar{x}}(-1)^{\bar{x}(\bar{a}+1)} zax = (-1)^{\bar{x}(\overline{za}+1)}(za)x$. Therefore, $za \in \mathcal{A}(\mathfrak{g})$, so $\overline{za} = \bar{0}$ by Lemma 3.1 again. This implies z is an even element.

(2) We know that $U(\mathfrak{g})$ is a free $\mathcal{Z}_0$-module with finite basis. Since $\mathcal{Z}_0$ is isomorphic to a polynomial ring, $U(\mathfrak{g})$ is a Noetherian $\mathcal{Z}_0$-module. Thus $\mathcal{Z}$ is finitely generated $\mathcal{Z}_0$-submodule of $U(\mathfrak{g})$. Assume $x_1, \cdots, x_l$ are a set of generators of $\mathcal{Z}$ over $\mathcal{Z}_0$. Then for each $z \in \mathcal{Z}$, we have equations with coefficients in $\mathcal{Z}_0$:

$$zx_i = \sum a_{ij} x_j, i = 1, \cdots, l.$$

And then, we have $\sum (z\delta_{ij} - a_{ij})x_j = 0, i = 1, \cdots, l$. Therefore, we have

$$(zI - A) \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = 0, \tag{3.1}$$

where I is the identity matrix, and $A = (a_{ij})_{l \times l}$. Multiplying the adjoint matrix of the coefficient matrix on the two sides of the equation (3.1), we can obtain

$$\det(zI - A) \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = 0.$$

This implies in force that $\det(zI - A) = 0$. Hence $\mathcal{Z}(\mathfrak{g})$ is integral over $\mathcal{Z}_0$. □

3.2. **The center does not contain any zero-divisors of $U(\mathfrak{g})$.** We will first show that $U(\mathfrak{g})$ is prime.

Definition 3.3. (cf. [4]) Let $L = L_{\bar{0}} + L_{\bar{1}}$ be a Lie superalgebra over k, and $\{y_1, y_2, \cdots, y_n\}$ be a basis of $L_{\bar{1}}$. In the symmetric algebra $\mathfrak{S}(L_{\bar{0}})$, we define an element $\mathsf{d}(L)$ which is the determinant of the $n \times n$ matrix over $\mathfrak{S}(L_{\bar{0}})$ whose (i, j)-entry is $x_{ij} := [y_i, y_j]$; if $L_{\bar{1}} = 0$, define $\mathsf{d}(L) = 1$.

Remark 3.4. Up to a nonzero scalar multiple, $\mathsf{d}(L)$ is not independent of the choice of basis. This is well-enough-defined for our purpose whether it is zero or not. Note also that $\mathsf{d}(L)$ is either 0 or a homogeneous polynomial in $\mathfrak{S}(L_{\bar{0}})$ of degree n.

[1]This statement can be proved directly by the triangular decomposition of basic classical Lie superalgebras. We thank Weiqiang Wang for his pointing it out.

The proof for Theorem 1.1 will be strongly dependent on the following result from [4].

Theorem 3.5. *(cf.* [4, Theorem 1.5]*) Let $L = L_{\bar 0} + L_{\bar 1}$ be a Lie superalgebra with a basis $\{y_1, \cdots, y_n\}$ of $L_{\bar 1}$, and with $\mathsf{d}(L) \in \mathfrak{S}(L_{\bar 0})$ defined as above. If $\mathsf{d}(L) \neq 0$, then $U(L)$ is prime.*

Turn to the basic classical Lie superalgebra $\mathfrak{g}$. Let's proceed our arguments for the proof of Theorem 1.1. Let $\Delta_{\bar 1}^+ = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ be the positive odd root space, where $r = \frac{1}{2}\dim \mathfrak{g}_{\bar 1}$, and e_{α_i} be the root vector of the one-dimensional root space $\mathfrak{g}_{\alpha_i}$ of α_i. Thus we have an ordered basis of $\mathfrak{g}_{\bar 1}$:

$$e_{\alpha_1}, e_{\alpha_2}, \cdots, e_{\alpha_r}, e_{-\alpha_r}, \cdots, e_{-\alpha_2}, e_{-\alpha_1}.$$

By Proposition 2.1, we have

$$[e_\alpha, e_\beta] = \begin{cases} a_{\alpha+\beta} e_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta \text{ and } \alpha+\beta \neq 0, \text{where } a_{\alpha+\beta} \in k; \\ 0, & \text{if } \alpha+\beta \notin \Delta; \\ (e_\alpha, e_{-\alpha}) h_\alpha, & \text{if } \alpha+\beta = 0. \end{cases}$$

In order to show that $\mathsf{d}(\mathfrak{g})$ is non-zero, it suffices to check that $\mathsf{d}(\mathfrak{g})$ is nonzero at some point as a polynomial in $\mathfrak{S}(\mathfrak{g}_{\bar 0})$. Note that all $x_{\alpha,\beta} := [e_\alpha, e_\beta]$ for $\alpha, \beta \in \Delta_1$ with $\alpha + \beta \in \Delta_0$, are algebraically independent of those nonzero $h \in \mathfrak{h}$ in $\mathfrak{S}(\mathfrak{g}_{\bar 0})$. So $\mathsf{d}(\mathfrak{g})$ is a polynomial over those nonzero elements $x_{\alpha,\beta}$ and h_α for $\alpha, \beta \in \Delta_1$ with $\alpha + \beta \in \Delta_0$. So we only need to make it certified that this polynomial is valued to be nonzero when we take those indeterminants to be some special values. For example, take all indetermiants $x_{\alpha,\beta}$ valued to be zero. This makes $\mathsf{d}(\mathfrak{g})$ take the following value:

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & \cdot & 0 & a_1 h_{\alpha_1} \\ 0 & 0 & \cdots & 0 & 0 & \cdot & a_2 h_{\alpha_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_r h_{\alpha_r} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & a_r h_{\alpha_r} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_2 h_{\alpha_2} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ a_1 h_{\alpha_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{vmatrix},$$

where $a_i = (e_{\alpha_i}, e_{-\alpha_i})$. This value is equal to $(-1)^r \prod_{i=1}^{r} a_i^2 h_{\alpha_i}^2$. By Proposition 2.1 we have $(e_{\alpha_i}, e_{-\alpha_i}) \neq 0$. Therefore $\mathsf{d}(\mathfrak{g}) \neq 0$. Hence $U(\mathfrak{g})$ is prime.

Thus we have the following corollary

Corollary 3.6. *Maintain the notations as in Theorem 1.1. Then $U(\mathfrak{g})$ is prime.*

Proof of Theorem 1.1(1): We will prove this statement by reduction to absurdity. Suppose there exists a zero-divisor $z \in \mathcal{Z}$ of $U(\mathfrak{g})$. Then there exists a nonzero element $u \in U(\mathfrak{g})$ such that $zu = 0$. Thus for any element $r \in U(\mathfrak{g})$, $zru = rzu = 0$, where the second equality is due to $\mathcal{Z} \subset U(\mathfrak{g})_{\bar 0}$. The primeness of

$U(\mathfrak{g})$ due to Corollary 3.6, implies in force that $z = 0$ or $u = 0$. Furthermore, it must happen that $z = 0$ because of the assumption $u \neq 0$, which contradicts to the hypothesis that z is a zero-divisor. So $\mathcal{Z}$ has no zero-divisor in $U(\mathfrak{g})$. We complete the proof.

In the sequel, we will always denote by $\mathbb{F}$ the fractional field of $\mathcal{Z}$. Set $\mathcal{D}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{\mathcal{Z}} \mathbb{F}$, which is a fractional ring of $U(\mathfrak{g})$ over $\mathbb{F}$.

3.3. **$\mathcal{D}(\mathfrak{g})$ is a simple superalgebra over the fractional filed $\mathbb{F}$ of $\mathcal{Z}$.** As we have shown that $\mathcal{Z} \subset U(\mathfrak{g})_{\bar{0}}$. Then it is obvious that $\mathcal{D}(\mathfrak{g})$ is isomorphic to the quotient algebra $U(\mathfrak{g})_{\mathcal{Z}^\times}$ where $\mathcal{Z}^\times = \mathcal{Z} \setminus \{0\}$ as an associative algebra.

Proof of Theorem 1.1(2): Recall that $U(\mathfrak{g})$ is a free $\mathcal{Z}_0$-module of rank $p^{\dim \mathfrak{g}_{\bar{0}}} 2^{\dim \mathfrak{g}_{\bar{1}}}$. So $U(\mathfrak{g})$ is a finitely generated $\mathcal{Z}$-module. By [24, Corollary 13.1.13], $U(\mathfrak{g})$ is a PI ring. Therefore $U(\mathfrak{g})_{\mathcal{Z}^\times}$ is a central simple algebra by Posner's theorem (cf. [24, Theorem 13.6.5]). Thus, $\mathcal{D}(\mathfrak{g})$ is a simple algebra over $\mathbb{F}$, as an ordinary associative algebra. And then it is naturally a simple superalgebra. We complete the proof.

Remark 3.7. By Wall's result (cf. [37, Theorem 1] or [20, Theorem 2.6]), $\mathcal{D}(\mathfrak{g})$ is isomorphic to $M(r|s)$ for some r, s. Here $M(r|s)$ stands for the k-algebra of square $(r+s)$-matrices with entries in k equipped with the gradings: $M(r|s)_{\bar{0}}$ = all square $(r+s)$-matrices for which $D = E = 0$ in the (r,s)-block form $\begin{pmatrix} C & D \\ E & F \end{pmatrix}$, and $M(r|s)_{\bar{1}}$ = all square $(r+s)$-matrices for which $D = E = 0$ in the above block form. Obviously, $M(r|s)$ is a simple superalgebra over k since the underlying algebra is simple. And, all simple (super) modules of $\mathcal{D}(\mathfrak{g})$ are isomorphic (cf. [20, Corollary 2.14].

4. Restriction homomorphisms

4.1. In the following we will always assume that G is a basic classical algebraic supergroup as described in 2.3, and assume $\mathfrak{g} = \mathrm{Lie}(G)$. Then the purely even subgroup G_{ev} is a reductive algebraic group, and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is naturally a basic classical Lie superalgebra with natural restricted structure which arises from $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_{\mathrm{ev}})$. Associated with a given Cartan subalgebra $\mathfrak{h}$, one has root space decomposition for $\mathfrak{g}_{\bar{0}}$ and for $\mathfrak{g}$ respectively:

$$\mathfrak{g}_{\bar{0}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta_0^-} \mathfrak{g}_\alpha$$

and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Recall that $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, $\Delta^- = \Delta_0^- \cup \Delta_1^-$, and $\Delta = \Delta^+ \cup \Delta^-$. One has a standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ for $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Set $\rho = \frac{1}{2}(\sum_{\alpha \in \Delta_0^+} \alpha - \sum_{\beta \in \Delta_1^+} \beta)$, and set $r = \dim \mathfrak{h}$.

Let W be the Weyl group of $\mathfrak{g}_{\bar{0}}$. Recall that W naturally acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively. Generally, for a W-set X we denote by $\mathbf{Z}_W(X)$ the pointwise stablizers of X, which is a normal subgroup of W. Set $\Omega = \{\chi \in \mathfrak{h}^* \subset \mathfrak{g}^* \mid \chi(h_\alpha) \neq 0, \alpha \in \Delta\}$, the set of regular semisimple elements over $\mathfrak{h}^*$. Denote by the quotient group $\overline{W} = W/\mathbf{Z}_W(\mathfrak{h})$, and $\Omega_1 := \{\chi \in \Omega \mid \bar{w}(\chi) = \chi, \forall\, \bar{w} \in \overline{W}\}$. Then $G_{\mathrm{ev}} \cdot \Omega_1$ contains a G_{ev}-stable open subset of $\mathfrak{g}_{\bar{0}}{}^*$ because Ω_1 is an open subset of $\mathfrak{h}^*$, and $G_{\mathrm{ev}} \cdot \mathfrak{h}^*$ contains a G_{ev}-stable open subset of $\mathfrak{g}_{\bar{0}}{}^*$. Elements of $G_{\mathrm{ev}} \cdot \Omega_1$ are called strongly regular. It's easily known that

$$W_\chi = \mathbf{Z}_W(\mathfrak{h}), \forall\, \chi \in \Omega_1 \tag{4.1}$$

where $W_\chi = \{w \in W \mid w(\chi) = \chi\}$, the stabilizer of χ in W.

Associated with $\chi \in \mathfrak{b}_{\bar{0}}^* \subset \mathfrak{g}_{\bar{0}}^*$ and $\lambda \in \Lambda(\chi) := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p\}$, one has a so-called baby Verma module $Z_\chi(\lambda)$ which is by definition an induced $U_\chi(\mathfrak{g})$-module arising from a one-dimensional module k_λ of $\mathfrak{b}$ presented by $(h+n)v_\lambda = \lambda(h)v_\lambda$, for $\forall h \in \mathfrak{h}, n \in \mathfrak{n}^+$, where v_λ is a basis vector of one-dimensional space k_λ. By the structure of $Z_\chi(\lambda)$, $Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} k_\lambda$ which coincides with $U_\chi(\mathfrak{n}^-) \otimes v_\lambda$, as k-spaces, where $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. So we have $\dim Z_\chi(\lambda) = p^{(\dim \mathfrak{g}_{\bar{0}} - r)/2} 2^{\dim \mathfrak{g}_{\bar{1}}/2}$. By Wang-Zhao's result, $Z_\chi(\lambda)$ is an irreducible $U_\chi(\mathfrak{g})$-module when χ is regular semisimple. And the set of p^r modules $\{Z_\chi(\lambda) \mid \lambda \in \Lambda(\chi)\}$ constitute the complete one of iso-classes of irreducible $U_\chi(\mathfrak{g})$-modules under such a circumstance (cf. [38, Theorem 5.6]).

Recall that the action of purely even subgroup G_{ev} on representations is given by $\Psi^g(x) = \Psi(g^{-1}xg)$, where $\Psi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a given representation of $\mathfrak{g}$. Let T be a maximal torus of G_{ev} with $\mathrm{Lie}(T) = \mathfrak{h}$. Then W can be identified with $N_{G_{\mathrm{ev}}}(T)/C_{G_{\mathrm{ev}}}(T)$ (cf. [18, §24.1]), and T stabilizes all points in Ω^p. Thus, the action of $N_{G_{\mathrm{ev}}}(T)$ factors through the action of W. So we can consider the action of the Weyl group $W = N_{G_{\mathrm{ev}}}(T)/C_{G_{\mathrm{ev}}}(T)$ on representations of $\mathfrak{g}$.

Simply write $Z_\chi(\lambda)$ as $V(\lambda)$ for a given regular semisimple p-character χ in the sequel argument. For emphasizing the dependence of $V(\lambda)$ on a Borel subalgebra, we shall write it by $V_{\mathfrak{b}}(\lambda)$ for a moment. By previous arguments, we can describe an action of $w \in W$ on baby Verma modules, by moving $V_{\mathfrak{b}}(\lambda)$ into $V_{\mathfrak{b}_w}(w(\lambda))$. Here $\mathfrak{b}_w$ means the Borel subalgebra $w \cdot \mathfrak{b} \cdot w^{-1}$ which equals to $\mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_{w(\alpha)}$. This is because if v is a highest vector in $V_{\mathfrak{b}}(\lambda)$ for $\mathfrak{b}$, then $(h+n)v = \lambda(h)v, h \in \mathfrak{h}, n \in \mathfrak{n}$. So, in $V_{\mathfrak{b}_w}(w(\lambda))$, $(w(h+n)w^{-1})v = \lambda(h)v$.

Note that for a representation Ψ of $\mathfrak{g}$ with a semi-regular p-character, we have $\Psi(e_\alpha)^p = 0$ for $\alpha \in \Delta_0$, and we have either $\Psi(e_\beta)^2 = 0$ for $\beta \in \Delta_1$ with $2\beta \notin \Delta$, or $\Psi(e_\beta)^2 \in k\Psi(e_{2\beta})$ for $\beta \in \Delta_1$ with $2\beta \in \Delta_0$. So $V_{\mathfrak{b}_w}(w(\lambda))$ admits a unique $\mathfrak{b}$-stable line. Thus, there must exist some $\lambda_w \in \mathfrak{h}^*$ such that $V_{\mathfrak{b}_w}(w(\lambda)) \cong V_{\mathfrak{b}}(\lambda_w)$.

We have the following Lemma, which will be important to Lemma 4.3.

Lemma 4.1. *Set* $\mathfrak{s}(w) = \sum_{\alpha \in \Delta_0(w)} \alpha - \sum_{\beta \in \Delta_1(w)} \beta$ *for* $w \in W$, *and* $\Delta(w) = \{\alpha \in \Delta^+ \mid w^{-1}\alpha \in \Delta^-\}$, $\Delta_i(w) = \Delta(w) \cap \Delta_i, i = 0, 1$. *Then the following statements hold.*

(1) $\lambda_w = w(\lambda) - \mathfrak{s}(w)$.
(2) $\mathfrak{s}(w) = \rho - w(\rho)$.

(3) *If* $w \in \boldsymbol{Z}_W(\mathfrak{h})$, *then* $\lambda_w = \lambda$.

Proof. The proof is an analogy of that for [23, Lemma 4.3]. We give a complete proof here for the readers' convenience because some new phenomena appear in the super case here, such as the different meaning of ρ.

Let v be a non-zero eigenvector for $\mathfrak{b}_w$ in $V_{\mathfrak{b}_w}(w(\lambda))$, on which the corresponding representation is denoted by Ψ temporally. Set

$$u = \prod_{\beta\in\Delta_1(w)} e_\beta \prod_{\alpha\in\Delta_0(w)} e_{\alpha_i}^{p-1}.$$

Then

$$\begin{aligned}\Psi^w(u)v &= (\prod_{\beta\in\Delta_1(w)} w^{-1}e_\beta w \prod_{\alpha\in\Delta_0(w)} w^{-1}e_\alpha^{p-1}w)v \\ &= C(\prod_{\beta\in\Delta_1(w)} e_{w^{-1}(\beta)} \prod_{\alpha\in\Delta_0(w)} e_{w^{-1}(\alpha)}^{p-1})v,\end{aligned}$$

which is denoted by v' with C a nonzero number in k. By definition, we know that v' is actually a nonzero vector of $V_{\mathfrak{b}_w}(w(\lambda))$. It's not difficult to check that v' is a nonzero $\mathfrak{b}$-eigenvector with highest weight $\lambda_w = w(\lambda) - \mathfrak{s}(w)$. Hence, Statement (1) is proved.

By induction beginning with $\mathfrak{s}(s_\alpha) = \rho - s_\alpha(\rho)$ we can easily prove (2). Combining (1) and (2), we get $\lambda_w = w(\lambda) + w(\rho) - \rho$. The condition $w \in \mathbf{Z}_W(\mathfrak{h})$ implies that $\lambda_w = \lambda$. $\square$

4.2. Next we consider the Harish-Chandra homomorphism $\gamma : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$, which is by definition the composite of the canonical projection $\gamma_1 : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ and the algebra homomorphism $\beta : U(\mathfrak{h}) \longrightarrow U(\mathfrak{h})$ defined via $h \mapsto h + \rho(h)$ for all $h \in \mathfrak{h}$. In the following arguments, we will identify $U(\mathfrak{h})$ (=the symmetric algebra $\mathfrak{S}(\mathfrak{h})$) with $k[\mathfrak{h}^*]$ the polynomial ring over $\mathfrak{h}^*$. The following lemma is an analogy of [21, Lemma 5.2], the proof of which can be done by the same arguments as in [21], omitted here.

Lemma 4.2. *(cf.* [21, Lemma 5.2]*) The restriction map of* γ*:* $U(\mathfrak{g})^T \longrightarrow U(\mathfrak{h})$ *is a homomorphism of algebras, still denoted by* γ*.*

Furthermore, we have the following

Lemma 4.3. *For* $w \in W$*,* $n_w \in N_{G_{ev}}(T)$ *a representative of* $w \in W$*, then we have* $\gamma(n_w z n_w^{-1}) = w\gamma(z)w^{-1}$*, for all* $z \in \mathcal{Z}$*. In particular,* $\gamma(\mathcal{Z}^{N_{G_{ev}}(T)}) \subseteq U(\mathfrak{h})^W$*.*

Proof. Maintain the notations as previously (in particular, as in the arguments prior to Lemma 4.1). Recall that for $u \in U(\mathfrak{g})$, the PBW theorem enables us to write uniquely $u = \varphi_0 + \sum u_i^- u_i^+ \varphi_i$, such that $\varphi_i \in U(\mathfrak{h}), u_i^\pm \in U(\mathfrak{n}^\pm)$, and $u_i^- u_i^+ \neq 0$. And then $\gamma_1(u) = \varphi_0$. And $\gamma = \beta \circ \gamma_1$. Recall that for $z \in \mathcal{Z}$, $z \in U(\mathfrak{g})_{\bar{0}}$ (cf. Lemma 3.2). Hence z acts as a scalar $\chi_\lambda(z)$ on the irreducible module $V_\mathfrak{b}(\lambda)$. In particular, $zv_\lambda = \chi_\lambda(z)v_\lambda$ for a highest weight vector v_λ in $V_\mathfrak{b}(\lambda)$. On the other hand, $zv_\lambda = \varphi_0(\lambda)v$. Here we identify $U(\mathfrak{h})$ with $\mathfrak{S}(\mathfrak{h})$. So $\chi_\lambda(z) = \varphi_0(\lambda) = \gamma_1(z)(\lambda)$. By Lemma 4.1, the image of $V_\mathfrak{b}(\lambda)$ under the action of w is isomorphic to $V_\mathfrak{b}(\lambda_w)$, and $n_w z n_w^{-1}$ acts on $V_\mathfrak{b}(\lambda_w)$ as a scalar $\chi_{\lambda_w}(z) =$

$\varphi_0(\lambda_w)$. Then, $\gamma(n_w z n_w^{-1})(\lambda) = \varphi_0(\lambda_w + \rho) = \varphi_0(w(\lambda+\rho)) = w\gamma(z)w^{-1}(\lambda)$. Hence $\gamma(n_w z n_w^{-1}) = w\gamma(z)w^{-1}$. □

4.3. We will still denote by $\mathrm{Frac}(R)$ the fractional filed of an integral domain R. We will consider the subalgebra $U(\mathfrak{g})^{G_{\mathrm{ev}}} \cap \mathcal{Z}$ which is just $\mathcal{Z}^{G_{\mathrm{ev}}}$, a large part of the whole center. But we remind the reader distinguishing the correspondent notation of Lie algebra. By Proposition 3.2, $\mathcal{Z}$ is integral over $\mathcal{Z}_0$. The following result is clear.

Lemma 4.4. *For each $z \in \mathcal{Z}^{G_{ev}}$, there exists a unique monic polynomial $g_0(x) \in$ $Frac(\mathcal{Z}_0)[x]$ with minimal degree among those polynomials taking $x = z$ to be a root.*

We can obtain the following proposition.

Proposition 4.5. *$\gamma : \mathcal{Z}^{G_{ev}} \longrightarrow U(\mathfrak{h})^W$ is an injective homomorphism of algebra.*

Proof. We have known that γ is an algebraic homomorphism by Lemma 4.2, and $\gamma(\mathcal{Z}^{G_{\mathrm{ev}}}) \subseteq U(\mathfrak{h})^W$ by Lemma 4.3. Therefore it is enough to verify that γ is injective. Take a nonzero element $z \in \mathcal{Z}^{G_{\mathrm{ev}}}$. We will prove that $\gamma(z) \neq 0$. According to Lemma 4.4, there exists a unique monic polynomial $g_0(x)$ with minimal degree such that $g_0(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1}z + a_m = 0, a_i \in \mathrm{Frac}(\mathcal{Z}_0)$. By [23, Lemma 4.6], we know $\mathrm{Frac}(\mathcal{Z}_0)^{G_{\mathrm{ev}}} = \mathrm{Frac}({\mathcal{Z}_0}^{G_{\mathrm{ev}}})$. So we immediately get that all those $a_i \in \mathrm{Frac}({\mathcal{Z}_0}^{G_{\mathrm{ev}}})$. For the restriction on ${\mathcal{Z}_0}^{G_{\mathrm{ev}}}$ of γ, we have known by the arguments in [23] that $\gamma : {\mathcal{Z}_0}^{G_{\mathrm{ev}}} \longrightarrow U(\mathfrak{h})^W$ is injective. By a natural extension, it is easy to check that γ is also injective on $\mathrm{Frac}(\mathcal{Z}_0^{G_{\mathrm{ev}}})$. Note that $\sigma(a) = a$ for $\sigma \in G_{\mathrm{ev}}$ and for $a \in \mathrm{Frac}(\mathcal{Z}_0)^{G_{\mathrm{ev}}} = \mathrm{Frac}({\mathcal{Z}_0}^{G_{\mathrm{ev}}})$. Applying the algebra homomorphism γ to that equation, we have $\sum\limits_{i=1}^{m} \gamma(a_{m-i})\gamma(z)^i + \gamma(a_m) = 0$. And then, $\gamma(z)(\sum\limits_{i=1}^{m} \gamma(a_{m-i})\gamma(z)^{i-1}) = -\gamma(a_m)$. By Theorem 1.1, $\mathcal{Z}^{G_{\mathrm{ev}}}$ has no zero-divisor. We claim that the constant term $a_m \neq 0$. Otherwise, $g_1(x) := x^{m-1} + a_1 x^{m-2} + \cdots + a_{m-1}$ is a non-zero polynomial with $g_1(z) = 0$, a contradiction with the assumption of minimal degree. This implies in force that $\gamma(a_m) \neq 0$. Hence, $\gamma(z) \neq 0$. □

5. The structure of center: proof of Theorem 1.2

Maintain the notations as before. In particular, set $\mathbb{F} := \mathrm{Frac}(\mathcal{Z})$ the fractional field of $\mathcal{Z} := \mathcal{Z}(\mathfrak{g})$. And set $\mathcal{D}(\mathfrak{g}) := U(\mathfrak{g}) \otimes_{\mathcal{Z}} \mathbb{F}$, which is a fractional ring of $U(\mathfrak{g})$ over $\mathbb{F}$.

5.1. By Proposition 3.2, $\mathcal{Z}$ is a finitely-generated commutative algebra over k. So we have the Zassenhaus variety $\mathrm{Maxspec}(\mathcal{Z})$ for $\mathfrak{g}$, analogous to the ordinary Lie algebra case, which is defined to be the maximal spectrum of $\mathcal{Z}$. We can identify $\mathrm{Maxspec}(\mathcal{Z})$ with *the affine algebraic variety of algebra morphisms $\mathcal{Z}$ to k, the latter of which will be denoted by $Spec(\mathcal{Z})$.*

Denote by *Spec*$(U(\mathfrak{g}))$ *the set of* G_{ev}*-equivalent irreducible* $U(\mathfrak{g})$*-module class.* Naturally, there is a G_{ev}-equivariant map

$$\pi : \mathrm{Spec}(U(\mathfrak{g})) \longrightarrow \mathrm{Spec}(\mathcal{Z}),$$

which is defined by the central characters over irreducible modules.

5.2. By Theorem 1.1, $\mathcal{D}(\mathfrak{g})$ is a finite-dimensional central simple algebra over $\mathbb{F}$. Set $q = \dim_{\mathbb{F}} \mathcal{D}(\mathfrak{g})$. Consider an irreducible $\mathcal{D}(\mathfrak{g})$-module $V_{\mathbb{F}}$ over $\mathbb{F}$. For a given non-zero vector $v \in V_{\mathbb{F}}$, we can write $V_{\mathbb{F}} = \mathcal{D}(\mathfrak{g})v$. Assume $\dim V_{\mathbb{F}} = l$, then $q = l^2$. Then we can assume that there are a set of $\mathbb{F}$-basis $\{v_1 = v, v_2, \cdots, v_l\}$ of $V_{\mathbb{F}}$, which are included in $U(\mathfrak{g})v$. By Proposition 3.2, $U(\mathfrak{g})$ is free over $\mathcal{Z}_0$ with finite basis. Then $U(\mathfrak{g})$ is a finitely-generated $\mathcal{Z}$-module. And we assume the $\{d_1, \cdots, d_m\}$ are a set of generators of $U(\mathfrak{g})$ over $\mathcal{Z}$. For each pair (d_i, v_j), there exists $z_{ij} \in \mathcal{Z}$, such that $z_{ij} d_i v_j \in \sum_t \mathcal{Z} v_t$. Denote by $\mathcal{C}$ the product of all the central elements z_{ij}. Then $V_{\mathcal{C}} := U(\mathfrak{g})\mathcal{C}v \subset \sum_i U(\mathfrak{g})\mathcal{C}v_i \subset \sum_t \mathcal{Z}v_t$.

For each $u \in U(\mathfrak{g})$, we have $u(\mathcal{C}v_1, \cdots, \mathcal{C}v_l) = (v_1, \cdots, v_l)U$, U is the matrix with each entry in $\mathcal{Z}$. So we can define $\Phi(u) = \mathrm{Tr}(U) \in \mathcal{Z}$ and the bilinear form $B(u_1, u_2) = \Phi(u_1 u_2)$.

Given $x_1, \cdots, x_q \in U(\mathfrak{g})$, consider qth determinant $\mathfrak{D} := \mathrm{Det}(B(x_i, x_j)_{q\times q})$, the ideal which generates in $\mathcal{Z}$ is called discriminant ideal. Recall that $\mathcal{D}(\mathfrak{g})$ is a central simple algebra over $\mathbb{F}$ of dimension q. There must exist a non-zero discriminant ideal in $\mathcal{Z}$. Naturally, $B(\cdot,\cdot)$ can be extended to a non-degenerate bilinear form in $\mathcal{D}(\mathfrak{g})$.

Definition 5.1. Define $\mathfrak{A} \subset \mathrm{Spec}(\mathcal{Z})$ to be the set $\{\varphi \in \mathrm{Spec}(\mathcal{Z}) \mid$ there exist $x_1, \cdots, x_q \in U(\mathfrak{g})$, such that $\mathfrak{D} = \mathrm{Det}(B(x_i, x_j)_{q\times q})$ satisfies $\varphi(\mathcal{C}\mathfrak{D}) \neq 0\}$.

By the arguments above, $\mathfrak{A}$ is a non-empty open subset of $\mathrm{Spec}(\mathcal{Z})$.

Lemma 5.2. *There exists an open dense subset* W *of* Spec$(\mathcal{Z})$*, such that the map* $\pi_W : \pi^{-1}W \longrightarrow W$ *induced from* π *is bijective.*

Proof. We will divide several steps to find a desirable W. Maintain the notations as in the arguments before Definition 5.1.

We first consider the fiber of π over φ for $\varphi \in \mathfrak{A}$. According to Definition of $\mathfrak{A}$, there exist $x_1, \cdots, x_q \in U(\mathfrak{g})$ such that $\varphi(\mathcal{C}\mathfrak{D}) \neq 0$, where $\mathfrak{D} = \mathrm{Det}(B(x_i, x_j)_{q\times q})$. Set $U_\varphi := U(\mathfrak{g}) \otimes_{(\mathcal{Z},\varphi)} k$, and $V_k = V_{\mathcal{C}} \otimes_{(\mathcal{Z},\varphi)} k$. Here the tensor products are defined over $\mathcal{Z}$, and k is regarded as a $\mathcal{Z}$-module induced from φ. Set $\bar{u}$ for $u \in U(\mathfrak{g})$ to be the image of u under the map $- \otimes_{(\mathcal{Z},\varphi)} 1 : U(\mathfrak{g}) \longrightarrow U_\varphi$.

We then claim that $\bar{x}_i, i = 1, \cdots, q$, form a basis of U_φ.

In fact, by the choice of $x_1, \cdots, x_q$, $\varphi(\mathfrak{D}) \neq 0$. So $x_i, i = 1, \cdots, q$ are $\mathbb{F}$-linear independent elements in $\mathcal{D}(\mathfrak{g})$. Then $x_i, i = 1, \cdots, q$ is a $\mathbb{F}$-basis of $\mathcal{D}(\mathfrak{g})$. As $B(\cdot,\cdot)$ is non-degenerate, we can take a dual basis $\{y_i, i = 1, \cdots, q\}$ of the basis $\{x_i, i = 1, \cdots, q\}$ in $\mathcal{D}(\mathfrak{g})$ with respect to this bilinear form. Thus, $\mathfrak{D}y_i \in U(\mathfrak{g})$. Furthermore, for each $u \in U(\mathfrak{g})$,

$$\mathfrak{D}u = \sum_{i=1}^{q} B(\mathfrak{D}u, y_i)x_i.$$

Since $\varphi(\mathfrak{D}) \neq 0$, and $B(\mathfrak{D}u, y_i) = B(u, \mathfrak{D}y_i) \in \mathcal{Z}$, we have $\bar{u} = \sum_{i=1}^{q} \varphi(\mathfrak{D})^{-1} \varphi(B(\mathfrak{D}u, y_i))\bar{x}_i$. That is to say, U_φ is k-spanned by $\{\bar{x}_i, i = 1, \cdots, q\}$. Furthermore, $\mathrm{Det}(B_\varphi(\bar{x}_i, \bar{x}_j)_{q\times q}) = \varphi(\mathfrak{D}) \neq 0$ where B_φ is k-valued of B through φ, B_φ then naturally becomes a non-degenerate trace form of U_φ defined by the module $V_k = V_{\mathcal{C}} \otimes_{(\mathcal{Z},\varphi)} k$. The claim is proved. Moreover, $\dim_k V_k = l$.

Next, we claim that U_φ is a semisimple algebra, naturally a semisimple superalgebra (cf. [11] or [20]). Note that $\mathcal{D}(\mathfrak{g})$ is a central simple algebra over $\mathbb{F}$. By the same arguments as used in the proof of [40, Theorem 5], we can get if $\bar{x}$ is a non-zero element in the Jacobson radical of U_φ, then for each $\bar{u} \in U_\varphi$, $B_\varphi(\bar{x}, \bar{u}) = 0$. The non-degeneracy of B_φ ensures that the Jacobson radical of U_φ is trivial. So the claim on the semisimple property of U_φ is proved.

Furthermore, we note that if the semisimple superalgebra U_φ is simple as an associative algebra, it has the form $\mathrm{Mat}_m(k)$. So $\rho : U(\mathfrak{g}) \longrightarrow \mathrm{Mat}_m(k)$ is a representation of $U(\mathfrak{g})$. And all irreducible representations of $\mathfrak{g}$ induced from φ in this way are isomorphic to V_k, naturally having the same central character.

Recall that $\dim_k U_\varphi = q = l^2$ by the arguments above, and that $\dim_k V_k = l$. To ensure that the semisimple superalgebra U_φ is simple, it is enough to ensure that V_k is a simple U_φ-module. For this, we take an irreducible submodule V_1 of V_k. Note that $V_k = U_\varphi v$. We can then take a nonzero element in V_1, written as fv with $f \in U_\varphi$. Then $V_1 = U_\varphi fv$. There exists $R \in U(\mathfrak{g})$, satisfying $\overline{R} = f$. Observe that for $\widetilde{V_1} := U(\mathfrak{g})R\mathcal{C}v \subset V_{\mathcal{C}}$, $\widetilde{V} := \widetilde{V_1} \otimes_{\mathcal{Z}} \mathbb{F}$ is a non-trivial submodule of $\mathcal{D}(\mathfrak{g})$ in $V_{\mathbb{F}}$. Hence, $\widetilde{V_1}$ and $V_{\mathcal{C}}$ are two $\mathcal{Z}$-lattices of $V_{\mathbb{F}}$. Since $V_{\mathbb{F}}$ is an irreducible $\mathcal{D}(\mathfrak{g})$-module over $\mathbb{F}$ and $\widetilde{V_1} \subset V_{\mathcal{C}}$, there must exist $P \in \mathcal{Z}$ such that $\widetilde{V_1} = PV_{\mathcal{C}}$. Hence $V_1 = \widetilde{V_1} \otimes_{(\mathcal{Z},\varphi)} k = PV_{\mathcal{C}} \otimes_{(\mathcal{Z},\varphi)} k$ (note that $\varphi(\mathcal{C})$ is a nonzero number in k and $\overline{R} = f \neq 0$). We get $\varphi(P) \neq 0$, hence $V_1 = V_k$. So, V_k is a simple $U(\mathfrak{g})$-module arising from φ, which is a unique simple module of the simple superalgebra U_φ, up to isomorphism (cf. Remark 3.7). Therefore, it suffices to take W equal to $\mathfrak{A}$. Then it is a desirable open set of $\mathrm{Spec}(\mathcal{Z})$. □

Remark 5.3. That $\mathcal{C}$ appearing in the first paragraph of §5.2 is related to the choice of irreducible $\mathcal{D}(\mathfrak{g})$-module $V_{\mathbb{F}}$. We can choose appropriate $V_{\mathbb{F}}$ such that $\mathcal{C} = 1$. In fact, $U(\mathfrak{g})$ itself is a $\mathcal{Z}$-lattice in the regular $\mathcal{D}(\mathfrak{g})$-module. Hence the central simple (super-) algebra $\mathcal{D}(\mathfrak{g})$ over $\mathbb{F}$ can be decomposed into the direct sum of some (left) regular irreducible supermodules as: $\mathcal{D}(\mathfrak{g}) = \bigoplus_i U_i$ (cf. [20, Proposition 2.11]). Then $U(\mathfrak{g}) \cap U_i$ becomes a $\mathcal{Z}$-lattice. In such a case, $\mathcal{C} = 1$.

5.3. We need to understand the field extension $\mathrm{Frac}(\mathcal{Z}) : \mathrm{Frac}(\mathcal{Z}_0)$. The following theorem is very important to the latter discussion.

Theorem 5.4. *(cf.* [32, Theorem 5.1.6]*) Let X and Y be irreducible varieties and let $\varphi : X \longrightarrow Y$ be a dominant morphism. Put $r = \dim X - \dim Y$, there exists non-empty open subset U of X with the following properties:*

(1) *Restricting on U, φ is open morphism $U \longrightarrow Y$.*
(2) *If Y' is the irreducible closed subvariety of Y, X' is the irreducible component of $\varphi^{-1}(Y')$ intersecting with U. Then $\dim X' = \dim Y' + r$. In*

particular, if $y \in Y$, any irreducible component of $\varphi^{-1}(y)$ that intersects U has dimensional r.

(3) *If $k(X)$ is algebraic over $k(Y)$, then for each $x \in U$, the dimension of the points in the fiber of $\varphi^{-1}(\varphi(x))$ is equal to the separable degree $[k(X), k(Y)]_s$ of $k(X)$ over $k(Y)$.*

Lemma 5.5. *Maintain the notations as previously. The separable degree of Frac($\mathcal{Z}$) over Frac($\mathcal{Z}_0$) is not smaller than p^r, $r = \dim \mathfrak{h}$.*

Proof. By Proposition 3.2, $\mathcal{Z}$ is integral over $\mathcal{Z}_0$. Thus we have a surjective map of irreducible varieties $\phi : \mathrm{Spec}(\mathcal{Z}) \longrightarrow \mathrm{Spec}(\mathcal{Z}_0)$, induced from the inclusion $\mathcal{Z}_0 \hookrightarrow \mathcal{Z}$ (cf. [29, ChV. §1. Ex.3] for surjective property). Consider the canonical G_{ev}-equivariant mapping sequence:

$$\mathrm{Spec}(U(\mathfrak{g})) \xrightarrow{\pi} \mathrm{Spec}(\mathcal{Z}) \xrightarrow{\phi} \mathrm{Spec}(\mathcal{Z}_0),$$

where $\pi : \mathrm{Spec}(U(\mathfrak{g})) \longrightarrow \mathrm{Spec}(\mathcal{Z})$ is defined from the central character associating to an irreducible representation. By lemma 5.2, π is generically bijective, that is, there exists an open dense subset W of $\mathrm{Spec}(\mathcal{Z})$ such that $\pi : \pi^{-1}(W) \longrightarrow W$ is bijective. Put $U = \phi^{-1}(G_{\mathrm{ev}} \cdot \Omega)$. Note that $G_{\mathrm{ev}} \cdot \Omega$ is an open set of $\mathfrak{g}_{\bar{0}}{}^*$, thereby regarded as an open set of $\mathrm{Spec}(\mathcal{Z}_0)$. Hence U becomes an open set of $\mathrm{Spec}(\mathcal{Z})$. Consider $S = U \cap W$. It is an open dense subset of $\mathrm{Spec}(\mathcal{Z})$. For $\chi \in G_{\mathrm{ev}} \cdot \Omega$, we might as well take $\chi \in \Omega$ without loss of generality. Set $\Lambda(\chi) = \{\lambda \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for } h \in \mathfrak{h}\}$. Recall that the set $\{Z_\chi(\lambda) \mid \lambda \in \Lambda(\chi)\}$ is just a complete set of G_{ev}-equivalent classes of irreducible modules (cf. [38, Corollary 5.7]). The number of the points in $\phi^{-1}(\chi)$ is bigger or equal to p^r. Since ϕ is finite dominant, by Theorem 5.4(3), the separable degree of $\mathrm{Frac}(\mathcal{Z})$ over $\mathrm{Frac}(\mathcal{Z}_0)$ is not smaller than p^r. □

Assume $\dim \mathfrak{g}_{\bar{0}} = s$, $\dim \mathfrak{g}_{\bar{1}} = t$. Then t is even in our case.

For the convenience of arguments in the sequel, we will continue to introduce the central-valued function Θ associated to a given pair (χ, λ) for $\mathcal{D}(\mathfrak{g})$ and for $U(\mathfrak{g})$, as we have done in the arguments used in Lemma 5.2, where $\chi \in \Omega$ and $\lambda \in \Lambda(\chi)$. This Θ is somewhat a concretization of the specialization introduced by Zassenhaus in [40, Page 25] which lays a foundation for the modular representation theory of Lie algebras.

Let's introduce Θ, which is by definition the canonical algebra homomorphism from $U(\mathfrak{g})$ to $U(\mathfrak{g})/(z - \Theta_{(\chi,\lambda)}(z))$, where $(z - \Theta_{(\chi,\lambda)}(z) \cdot 1)$ means the ideal of $U(\mathfrak{g})$ generated by z taking all through $\mathcal{Z}$, and $\Theta_{(\chi,\lambda)}(z)$ is defined by $z \cdot v_\lambda = \Theta_{(\chi,\lambda)}(z) v_\lambda$, where v_λ is the canonical generator of the baby Verma module $Z_\chi(\lambda)$. The category of $\Theta(U(\mathfrak{g}))$-modules is a subcategory of $U_\chi(\mathfrak{g})$-modules.

Proposition 5.6. *Frac($\mathcal{Z}$) is separable over Frac($\mathcal{Z}_0$), and [Frac($\mathcal{Z}$) : Frac($\mathcal{Z}_0$)] $= p^r, r = \dim \mathfrak{h}$.*

Proof. On one side, we fix $\chi \in \Omega$, a regular semisimple p-character. Then, associated with χ the baby Verma module $Z_\chi(\lambda)$ is an irreducible $U_\chi(\mathfrak{g})$-module (cf. [38,

Corollary 5.7]), and

$$\dim Z_\chi(\lambda) = p^{(\dim \mathfrak{g}_{\bar{0}}-r)/2} 2^{\dim \mathfrak{g}_{\bar{1}}/2} = p^{s-r/2} 2^{t/2}. \tag{5.1}$$

On the other side, we recall that $\mathcal{D}(\mathfrak{g})$ is a central simple $\mathbb{F}$-algebra, all simple $\mathcal{D}(\mathfrak{g})$-modules are isomorphic (cf. Remark 3.7). Still assume that $V_{\mathbb{F}}$ is a simple $\mathcal{D}(\mathfrak{g})$-module with dimension l, which admits a $\mathcal{Z}$-lattice $V = U(\mathfrak{g})v$ (cf. Remark 5.3). We may write $V_{\mathbb{F}} = \mathcal{D}(\mathfrak{g})v$. Thus, given k-valued Θ for $\mathcal{Z}$ as previously, we can take $V_k = \Theta(U(\mathfrak{g}))v$. In this way, we actually have had a module-operator $\Theta^V : V \longrightarrow V_k$, satisfying $\Theta^V(uw) = \Theta(u)\Theta^V(w)$, $\Theta^V(v) = v$, for each $u \in U(\mathfrak{g})$, $w \in V$.

Recall that as in Remark 5.3, $\mathcal{D}(\mathfrak{g})$ is the sum of isomorphic simple left modules, $\mathcal{D}(\mathfrak{g}) = \sum_{i=1}^{l} V_i$, $V_i = \mathcal{D}(\mathfrak{g})u_i$, and $V_{\mathbb{F}}$ is isomorphic to each V_i. In the following arguments, we might as well assume $V_{\mathbb{F}} = V_1$. Thus, $\Theta(u_1) \neq 0$, and $V_k = \Theta(U(\mathfrak{g})u_1)$. Next we claim the dimension inequality:

$$\dim_{\mathbb{F}} \mathcal{D}(\mathfrak{g}) \geq \dim_k \Theta(U(\mathfrak{g})) \geq \dim_k \mathrm{End_k(V_k)} \geq (\mathrm{dim_k}\, \mathrm{Z}_\chi(\lambda))^2. \tag{5.2}$$

Let's prove it below. The first inequality is obvious. As for the third inequality, it is because that V_k is a $U_\chi(\mathfrak{g})$-module, and $Z_\chi(\lambda)$ is an irreducible $U_\chi(\mathfrak{g})$-module, while all irreducible $U_\chi(\mathfrak{g})$-modules have the same dimension. Hence, it is enough to prove the second inequality. For this we only need to verify that V_k is an irreducible $\Theta(U(\mathfrak{g}))$-module. If this is not true, then V_k must contain a proper irreducible submodule W which is of form $\Theta(U(\mathfrak{g}))\Theta^V(uu_1)$. Put $U' = U(\mathfrak{g})uu_1$. Then $W = \Theta(U')$. Noticing that in $\mathcal{D}(\mathfrak{g})$, the left ideal U' must be a direct sum of some simple left ideals (cf. [20, Proposition 2.11]), we may assume that U' contains the summation $U(\mathfrak{g})u_i$ which naturally satisfies $\Theta(u_i) \neq 0$. Under the action of Θ, $\Theta(U(\mathfrak{g})u_i)$ is isomorphic to $V_k = \Theta(U(\mathfrak{g})u_1)$, as $\Theta(U(\mathfrak{g}))$-modules. This gives rise to a contradiction: $V_1 \supsetneqq W = \Theta(U') \supset \Theta(U(\mathfrak{g})u_i) \cong V_1$. Thus, we complete the proof of (5.2).

Recall $(\dim Z_\chi(\lambda))^2 = p^{s-r}2^t$. We then have $\dim_{\mathbb{F}} \mathcal{D}(\mathfrak{g}) \geq p^{s-r}2^t$. Therefore, $[\mathrm{Frac}(\mathcal{Z}) : \mathrm{Frac}(\mathcal{Z}_0)] \leq \dim_{\mathrm{Frac}(\mathcal{Z}_0)} \mathcal{D}(\mathfrak{g}) / \dim_{\mathbb{F}} \mathcal{D}(\mathfrak{g}) \leq p^r$. Combined with Lemma 5.5, $[\mathrm{Frac}(\mathcal{Z}) : \mathrm{Frac}(\mathcal{Z}_0)] = p^r = [\mathrm{Frac}(\mathcal{Z}) : \mathrm{Frac}(\mathcal{Z}_0)]_s$. Hence, the inequalities in (5.2) turn out to be real equalities. □

5.4. For the further arguments, we need some preparation involving (geometric) quotient spaces (to see [3], [9] and [26] for more details).

Definition 5.7. (cf. [9, §II.6.1]) Let $\zeta : V \longrightarrow W$ be a k-morphism of k-varieties. We say ζ is a *quotient morphism* if the following items satisfy:

(1) ζ is surjective and open.
(2) If $U \subset V$ is open, then ζ_* induces an isomorphism from $k[\zeta(U)]$ onto the set of $f \in k[U]$ which are constant on the fibers of $\zeta|_U$.

Definition 5.8. (cf. [9, §II.6.3]) We fix a k-group H acting k-morphically on a variety V.

(1) An orbit map is a surjective morphism $\zeta : V \longrightarrow W$ of varieties such that the fibers of ζ are the orbits of G in V.
(2) *A (geometric) quotient* of V by H over k is an orbit map $\zeta : V \longrightarrow W$ which is a quotient morphism over k in the sense of Definition 5.7. In this case, W is called *the (geometric) quotient space of V by H, denoted by V/H.*

Now we have a key lemma for our main theorem.

Lemma 5.9. $[Frac(\mathcal{Z})^{G_{ev}} : Frac(\mathcal{Z}_0)^{G_{ev}}] = p^r$.

Proof. The proof is a "super" analogy of [23, Lemma 4.4]. We still give the details for the readers' convenience.

By Proposition 5.6, $\mathrm{Frac}(\mathcal{Z})$ is a separable extension over $\mathrm{Frac}(\mathcal{Z}_0)$, and the separable degree is p^r. Consider the surjective G_{ev}-equivariant map of varieties $\phi : \mathrm{Spec}(\mathcal{Z}) \longrightarrow \mathrm{Spec}(\mathcal{Z}_0)$ induced from the algebraic embedding $\mathcal{Z}_0 \hookrightarrow \mathcal{Z}$. Recall that $\mathrm{Spec}(\mathcal{Z}_0)$ can be regarded as $\mathfrak{g}_{\bar{0}}^{*(1)}$. By Rosenlicht's theorem (cf. [27]), $\mathfrak{g}_{\bar{0}}^*$ has a G_{ev}-stable open dense subset V such that the quotient space of V by G_{ev} can be defined. Since $V_0 := G_{\mathrm{ev}} \cdot \Omega_1$ is an open set of $\mathfrak{g}_{\bar{0}}^*$, then $U = V \cap V_0$ is a G_{ev}-stable open set of irreducible variety $\mathfrak{g}_{\bar{0}}^*$. Moreover by the property of geometric quotients, we have $k(U/G_{\mathrm{ev}}) = \mathrm{Frac}(\mathcal{Z}_0)^{G_{\mathrm{ev}}}$, where the geometric quotient U/G_{ev} is the G_{ev}-orbit space of U. Set $W = \phi^{-1}(U)$, a G_{ev}-stable open set of $\mathrm{Spec}(\mathcal{Z})$. Note that the stabilizer $(G_{\mathrm{ev}})_\chi$ of χ in G_{ev} acts trivially on $\phi^{-1}(\chi)$, for $\chi \in U$. Hence, the geometric quotient W/G_{ev} exists, which then induces naturally a morphism of orbit spaces

$$\phi_{G_{\mathrm{ev}}} : W/G_{\mathrm{ev}} \longrightarrow U/G_{\mathrm{ev}}.$$

Comparing Theorem 5.4(3), we know that $\phi_{G_{\mathrm{ev}}}$ is still a separable morphism with degree p^r. The proposition follows. □

The following lemma is for ordinary Lie algebra case.

Lemma 5.10. *(cf.* [23, Lemma 4.6]*)* $Frac(\mathcal{Z}_0)^{G_{ev}} = Frac({\mathcal{Z}_0}^{G_{ev}})$.

Then we will have the following super version of Lemma 5.10.

Proposition 5.11. $Frac(\mathcal{Z})^{G_{ev}} = Frac(\mathcal{Z}^{G_{ev}})$.

Before the proof, we need the following result.

Theorem 5.12. *(cf.* [3, I.5.3]*) Let X be an affine algebraic variety with an action of an affine reductive algebraic group. Then there exists a unique (up to an isomorphism) algebraic variety Y such that $k[Y] = k[X]^G$, The variety Y is called the affine quotient space of X by the action of G. The inclusion $k[X]^G \subset k[X]$ induces a morphism $\zeta : X \longrightarrow Y$. The morphism ζ is called the affine quotient morphism of X by the action of G.*

Proof of Proposition 5.11: By Hilbert-Nagata Theorem, $\mathcal{Z}^{G_{\mathrm{ev}}}$ is finitely generated (cf. [26, Theorem A.1.0]). Set $X = \mathrm{Spec}(\mathcal{Z})$, $Y = \mathrm{Spec}(\mathcal{Z}^{G_{\mathrm{ev}}})$, denote by $\zeta : X \longrightarrow Y$, the morphism of irreducible varieties induced from the inclusion

$\mathcal{Z}^{G_{\mathrm{ev}}} \hookrightarrow \mathcal{Z}$. By Theorem 5.12, ζ satisfies the property of affine quotient map. Then ζ is submersive (see [26, Theorem A.1.1] or [3, Theorem I.5.4]). Moreover, by [26, Proposition 1.9], there exists a maximal G_{ev}-stable open subset X' of X such that ζ restricting on X' is geometric quotient map. Let $Y' = \zeta(X')$, then $X' = \zeta^{-1}(Y')$ (see [26, Theorem A.1.1] or [3, Theorem I.5.4]). Hence Y' is an open set of Y. From the irreducibility of X, Y, it follows that $\mathrm{Frac}(k[X]) = \mathrm{Frac}(k[X'])$ and $\mathrm{Frac}(k[Y]) = \mathrm{Frac}(k[Y'])$. Again by [9, Proposition II.6.5], $\mathrm{Frac}(k[X'])^{G_{\mathrm{ev}}} \cong \mathrm{Frac}(k[Y'])$. Therefore, $\mathrm{Frac}(\mathcal{Z})^{G_{\mathrm{ev}}} = \mathrm{Frac}(\mathcal{Z}^{G_{\mathrm{ev}}})$. We complete the proof.

Remark 5.13. We have a more general result than the above proposition.

G**-quotient Lemma.** *Let k be a field with prime characteristic, and A be a finitely generate integral k-ring. If G is a reductive algebraic group or finite group of A acting on A as automorphism of A, then $\mathrm{Frac}(A)^G = \mathrm{Frac}(A^G)$.*

Proof. When G is a reductive algebraic group, the proof can be seen in Proposition 5.11. For the case of finite group G, by Deligne's theorem (see [3, Theorem I.4.3.2]), the canonical affine quotient map $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A^G)$ is a geometric quotient map. The lemma follows. $\square$

5.5. **Proof of Theorem 1.2**(1): According to [23, Lemma 5.4] and its proof, we have known that $U(\mathfrak{h})^W$ is integral over $(U(\mathfrak{h})\cap\mathcal{Z}_0)^W$, $[\mathrm{Frac}(U(\mathfrak{h})^W) : \mathrm{Frac}((U(\mathfrak{h})\cap\mathcal{Z}_0)^W)] \leq p^r$, and that $\gamma({\mathcal{Z}_0}^{G_{\mathrm{ev}}}) = (U(\mathfrak{h}) \cap \mathcal{Z}_0)^W$. Taking Proposition 4.5 into account, we consider the following computation:

$$
\begin{aligned}
& [\mathrm{Frac}(U(\mathfrak{h})^W) : \mathrm{Frac}((U(\mathfrak{h}) \cap \mathcal{Z}_0)^W)] \\
= \ & [\mathrm{Frac}(U(\mathfrak{h})^W) : \mathrm{Frac}(\gamma({\mathcal{Z}_0}^{G_{\mathrm{ev}}}))] \\
= \ & [\mathrm{Frac}(U(\mathfrak{h})^W) : \mathrm{Frac}(\gamma(\mathcal{Z}^{G_{\mathrm{ev}}}))][\mathrm{Frac}(\gamma(\mathcal{Z}^{G_{\mathrm{ev}}})) : \mathrm{Frac}(\gamma({\mathcal{Z}_0}^{G_{\mathrm{ev}}}))] \\
= \ & [\mathrm{Frac}(U(\mathfrak{h})^W) : \mathrm{Frac}(\gamma(\mathcal{Z}^{G_{\mathrm{ev}}}))][\mathrm{Frac}(\mathcal{Z}^{G_{\mathrm{ev}}}) : \mathrm{Frac}({\mathcal{Z}_0}^{G_{\mathrm{ev}}})].
\end{aligned}
$$

By Lemma 5.10, Proposition 5.11, and Lemma 5.9, we have

$$[\mathrm{Frac}(\mathcal{Z}^{G_{\mathrm{ev}}}) : \mathrm{Frac}({\mathcal{Z}_0}^{G_{\mathrm{ev}}})] = [\mathrm{Frac}(\mathcal{Z})^{G_{\mathrm{ev}}} : \mathrm{Frac}(\mathcal{Z}_0)^{G_{\mathrm{ev}}}] = p^r.$$

Hence it follows in force that $\mathrm{Frac}(U(\mathfrak{h})^W) = \mathrm{Frac}(\gamma(\mathcal{Z}^{G_{\mathrm{ev}}}))$. Thus we complete the proof.

Remark 5.14. We remind the readers to compare the result with the one for the complex Lie superalgebras (cf. [22, Lemmas 1-2]).

Proof of Theorem 1.2(2): Consider the inclusion $\mathcal{Z}_0 \hookrightarrow \mathcal{Z}_1 \hookrightarrow \mathcal{Z}$, and $\mathcal{Z}^{G_{\mathrm{ev}}} \hookrightarrow \mathcal{Z}_1 \hookrightarrow \mathcal{Z}$. There is a commutative diagram of dominant morphism of irreducible

varieties

$$\begin{tikzcd}
 & & \mathrm{Spec}(\mathcal{Z}_0) \\
\mathrm{Spec}(\mathcal{Z}) \arrow[r, "\varphi_1"] \arrow[urr, "f_3"] \arrow[drr, "f_1"'] & \mathrm{Spec}(\mathcal{Z}_1) \arrow[ur, "\varphi_2"'] \arrow[dr, "f_2"] & \\
 & & \mathrm{Spec}(\mathcal{Z}^{G_{\mathrm{ev}}}).
\end{tikzcd}$$

By the proof of Proposition 5.11, there exist a G_{ev}-stable nonempty open subset V of $\mathrm{Spec}(\mathcal{Z})$, and a G_{ev}-stable nonempty open subset W of $\mathrm{Spec}(\mathcal{Z}^{G_{\mathrm{ev}}})$ such that $f_1|_V : V \longrightarrow W$ is a geometric quotient map. So, for a $w \in W$, all elements of $f_1^{-1}(w)$ fall in the same G_{ev}-orbit. Recall that $G_{\mathrm{ev}} \cdot \Omega_1$ contains a G_{ev}-stable open dense subset of $\mathrm{Spec}(\mathcal{Z}_0)$, say U_1. Set $U = U_1 \cap f_2^{-1}(W)$. By the irreducibility of $\mathrm{Spec}(\mathcal{Z}_1)$ and of $\mathrm{Spec}(\mathcal{Z}^{G_{\mathrm{ev}}})$, and by the dominance of f_2, we have that U is a G_{ev}-stable open dense subset of $\mathrm{Spec}(\mathcal{Z}_1)$.

Take $x \in U$. By Proposition 3.2(2) and Proposition 5.6, both φ_1 and φ_2 are finite separable morphisms. We assume that $\varphi_1^{-1}(x) = \{x_1, \cdots, x_d\}$. We assert that $d = 1$. Actually, by the choice of x there exists $w \in W$ such that $f_2(x) = w$. Then $\varphi_1^{-1}(x) \subseteq \varphi_1^{-1}(f_2^{-1}(w)) \subseteq (f_2 \circ \varphi_1)^{-1}(w) = f_1^{-1}(w)$. According to the above arguments and the property of geometric quotient, $x_1, \cdots, x_t$ are contained in one orbit. Now $\mathcal{Z}^{G_{\mathrm{ev}}} \subset \mathcal{Z}_1$, hence the stabilizer $(G_{\mathrm{ev}})_x$ of x in G_{ev} permutes them transitively. Set $y = \varphi_2(x)$. Then $y \in G_{\mathrm{ev}} \cdot \Omega_1$, and $(G_{\mathrm{ev}})_x \subseteq (G_{\mathrm{ev}})_y$. Since by Lemma 4.1(3) and Equation (4.1), $(G_{\mathrm{ev}})_y$ acts trivially on the fiber over y of $(\varphi_2\varphi_1)^{-1}$, the same is true for $(G_{\mathrm{ev}})_x$. Hence $d = 1$ as we asserted. Furthermore, since a geometric quotient map is submersive, two open sets $\phi_1^{-1}(U)$ and U are isomorphic. Therefore, we get that $\mathrm{Spec}(\mathcal{Z})$ and $\mathrm{Spec}(\mathcal{Z}_1)$ are birational equivalent. The proof is completed.

Remark 5.15. There are some natural questions on the Zassenhaus variety $\mathrm{Maxspec}(\mathcal{Z})$ remaining:

(1) In which case is the Zassenhaus variety $\mathrm{Maxspec}(\mathcal{Z})$ normal?
(2) Then how to describe the smooth points?
(3) How do the geometric properties of the Zassenhaus variety reflect the representation theory of $\mathfrak{g}$? For instance, whether the locus of smooth points coincide with the Azumaya locus which reflects the irreducible modules of maximal dimension in the modular representations of usual classical Lie algebras as shown in [6]?

We will make some investigation on these issues for $\mathfrak{g} = \mathfrak{osp}(1|2)$ next sections.

6. The further study of $\mathcal{Z}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{osp}(1|2n)$

Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ in this section. We will show that the homomorphism γ defined in Proposition 4.5 is an algebra isomorphism, and furthermore $\mathcal{Z}$ coincides with $\mathcal{Z}_1$ appearing in Theorem 1.2 in such a case.

6.1. Let $\{x_1, \cdots, x_s\}$ be a basis of $\mathfrak{g}_{\bar{0}}$, and $\{y_1, \cdots, y_t\}$ a basis of $\mathfrak{g}_{\bar{1}}$. Then $U(\mathfrak{g})$ has a PBW basis consisting of elements of the form $x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t}$, for non-negative integers a_i , and $b_i = 0$ or 1.

By a direct verification, one can know that $\mathfrak{g}$ satisfies a so-called absolutely torsion-free condition: $[y, y] \neq 0$ for any nonzero element $y \in \mathfrak{g}_{\bar{1}}$. So the universal enveloping superalgebra $U(\mathfrak{g})$ has the following featured properties.

Lemma 6.1. *Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. The following statements hold.*

(1) *$U(\mathfrak{g})$ has no zero-divisor.*
(2) *The global dimension of $U(\mathfrak{g})$ is finite.*

Proof. (1) It follows from a criterion due to Aubry-Lemaire, judging the existence of zero-divisors for $U(\mathfrak{g})$ (cf. [1]).

(2) It's a direct consequence of [8, Theorem 2]. □

There is a canonical filtration on $U(\mathfrak{g})$. Let $U^0 = k, U^1 = k + \mathfrak{g}$, and U^n be the subspace of $U(\mathfrak{g})$ spanned by all monomials of degree less than or equal to n. Here we take the $x_i's$ and the $y_i's$ to be of degree one. The associated graded ring $\mathrm{Gr}(U(\mathfrak{g})) = \Lambda_{\mathcal{R}}(y_1, \cdots, y_t)$ is an exterior algebra in $y_1, \cdots, y_t$ over the polynomial ring $\mathcal{R} := k[x_1, \cdots, x_s]$. So $\mathrm{Gr}(U(\mathfrak{g}))$ is a Noetherian ring. Therefore, $U(\mathfrak{g})$ is Noetherian. Furthermore, by [24, Corollary 13.1.13] we know that $U(\mathfrak{g})$ is a PI ring. The statements here are true for any finite-dimensional Lie superalgebras over k.

Recall that a ring R is called Auslander-Gorenstein if R is a Noetherian ring of finite (right and left) injective dimension with the additional property that for every finitely generated R-module M and every submodule $N \subseteq \mathrm{Ext}_R^j(M, R)$, one has $\mathrm{Ext}_R^i(N, R) = 0 \ \forall\ i < j$. If furthermore R is of finite (right and left) global dimension, we call it Auslander-regular. Let $j(M) = \min\{j : \mathrm{Ext}_R^j(M, R) \neq 0\}$, then R is called Cohen-Macaulay provided that $\mathrm{GK}\dim(R) < \infty$, and $j(M) + \mathrm{GK}\dim(M) = \mathrm{GK}\dim(R)$ holds for every finitely generated R-module M. If the analogous properties for R in the above holds when the GK dimension is replaced by Krull dimension, then we call R Macaulay. We have the following Lemma.

Lemma 6.2. *Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. Then the enveloping superalgebra $U(\mathfrak{g})$ is an Auslander-regular, Cohen-Macaulay ring.*

Proof. By Lemma 6.1(2), it suffices to show $U(\mathfrak{g})$ is an Auslander-Gorenstein, Cohen-Macaulay ring. By the arguments in the paragraph concerning the canonical filtration of $U(\mathfrak{g})$, the associated graded ring $\mathrm{Gr}(U(\mathfrak{g}))$ is a Noetherian PI ring of finite injective dimension. The lemma follows from [33, Corollary 4.5]. □

Proposition 6.3. *Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. The center $\mathcal{Z}$ of $U(\mathfrak{g})$ is integrally closed, i.e. the affine variety Maxspec($\mathcal{Z}$) is normal.*

Proof. This is an application of [33, Theorem 5.4(iii) and Lemma 4.3], together with Lemmas 6.1 and 6.2. □

Hence we have the following result.

Corollary 6.4. *$\mathcal{Z}^{G_{ev}}$ is integrally closed.*

6.2. **The structure of $\mathcal{Z}(\mathfrak{g})$.**

Theorem 6.5. *Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. Maintain the notations as in Theorems 1.1 and 1.2. Then the following statements hold.*

(1) $\gamma : \mathcal{Z}^{G_{ev}} \longrightarrow U(\mathfrak{h})^W$ *is an isomorphism.*
(2) $\mathcal{Z} = \mathcal{Z}_1$.

Proof. (1) On one hand, γ is injective (Proposition 4.5). On the other hand, $\mathrm{Frac}(\gamma(\mathcal{Z}^{G_{ev}})) = \mathrm{Frac}(U(\mathfrak{h})^W)$ (Theorem 1.2). By Corollary 6.4, $\gamma(\mathcal{Z}^{G_{ev}}) = U(\mathfrak{h})^W$. So, γ is an isomorphism.

(2) According to Theorem 1.2, $\mathrm{Frac}(\mathcal{Z}_1) = \mathrm{Frac}(\mathcal{Z})$. And $\mathcal{Z}$ is integrally closed (Proposition 6.3). So it suffices to prove $\mathcal{Z}_1$ is integrally closed. For this purpose, we only need to show the algebra $\mathcal{Z}_0 \otimes_{\mathcal{Z}_0{}^{G_{ev}}} \mathcal{Z}^{G_{ev}}$ is integrally closed.

Notice that $\mathcal{Z}^{G_{ev}}$ is naturally a ${\mathcal{Z}_0}^{G_{ev}}$-module. We have $\mathcal{Z}_0 \otimes_{\mathcal{Z}_0{}^{G_{ev}}} \mathcal{Z}^{G_{ev}}$ is isomorphic to $\mathcal{Z}_0 \otimes_{\mathcal{Z}_0{}^{G_{ev}}} U(\mathfrak{h})^W$, as algebras. As W is the Weyl group of $\mathfrak{g}_{\bar{0}}$, a classical fact reveals that $\mathcal{Z}_0 \otimes_{\mathcal{Z}_0{}^{G_{ev}}} U(\mathfrak{h})^W$ is isomorphic to $\mathcal{Z}(\mathfrak{g}_{\bar{0}})$ as algebras (cf. [36, 3.1], [23, 4.7], [6, 3.5(5)], etc.). Here $\mathcal{Z}(\mathfrak{g}_{\bar{0}})$ is the center of $U(\mathfrak{g}_{\bar{0}})$. As a well-known fact, $\mathcal{Z}(\mathfrak{g}_{\bar{0}})$ is integrally closed (cf. [40, Lemma 6] or [34, 6.5.4]), so is $\mathcal{Z}_0 \otimes_{\mathcal{Z}_0{}^{G_{ev}}} \mathcal{Z}^{G_{ev}}$. Thus we finish the proof. $\square$

The following result is a direct consequence.

Corollary 6.6. *As algebras, $\mathcal{Z} \cong \mathcal{Z}(\mathfrak{g}_{\bar{0}})$.*

7. Azumaya locus and smooth locus of Maxspec($\mathcal{Z}(\mathfrak{g})$) for $\mathfrak{g} = \mathfrak{osp}(1|2)$

From the arguments in the previous section, we know that $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ has the same geometrical properties as $\mathcal{Z}(\mathfrak{g}_{\bar{0}})$ for $\mathfrak{g} = \mathfrak{osp}(1|2n)$. In the concluding section, we will demonstrate the close relation between the representations of $U(\mathfrak{g})$ and the geometry of Maxspec($\mathcal{Z}(\mathfrak{g})$) for $\mathfrak{g} = \mathfrak{osp}(1|2)$. However, it's not clear which points in Maxspec($\mathcal{Z}$) are smooth, and what the precise relation is between the representations and the Azumaya property for $n > 2$. Those will be some topics in the future investigation.

We begin our arguments with some understanding on super representations. Let $A = A_{\bar{0}} + A_{\bar{1}}$ be a superalgebra over k. Denote by A-**smod** the category of finite-dimensional A-supermodules, and by A-**emod** the underlying even category of A-**smod** where we take the same objects of A-**smod** but only the even homomorphisms.

7.1. **The categories A-smod and A-emod.**

A superalgebra analogue of Schur Lemma states that the endomorphism ring of an irreducible module in A-**smod** is either one-dimensional or two-dimensional (cf. [11, 2.4]). An irreducible module is said to be of type M (resp. type Q) if its endomorphism ring is one-dimensional (resp. two-dimensional).

For any A-module M, there is a new module defined by exchanging the $\mathbb{Z}_2$-grading, which we denote by ΠM. It is easy to show that M is irreducible if and

only if ΠM is irreducible. Let $s = \dim M_{\bar{0}}$, $t = \dim M_{\bar{1}}$, and $u_1, \cdots, u_s$ (resp. $v_1, \cdots, v_t$) be a basis of $M_{\bar{0}}$ (resp. $M_{\bar{1}}$). Define $f : M \longrightarrow \Pi M$ by

$$f\left(\sum a_i u_i + \sum b_j v_j\right) = \sum a_i u_i - \sum b_j v_j.$$

It is easy to see that f is an isomorphism of A-modules of degree 1.

Lemma 7.1. *Assume the number of isomorphism classes of irreducible modules in A-**smod** is finite. Let $\{M_1, \cdots, M_l\}$ (resp. $\{N_1 \cdots, N_q\}$) be all non-isomorphic irreducible modules of type M (resp. of type Q) in A-**smod**. Then*

$$\{M_1, \cdots, M_l, \Pi M_1, \cdots, \Pi M_l, N_1, \cdots, N_q\}$$

*are all isomorphism classes of irreducible modules in A-**emod**.*

Proof. There is neither even isomorphism between ΠM_i and ΠM_j, nor one between ΠN_i and ΠN_j, $\forall i \neq j$ in A-**smod**. Otherwise, we could get an isomorphism between M_i and M_j, or between N_i and N_j. Similarly, there is neither even isomorphism between M_i and ΠM_j nor one between N_i and ΠN_i for $i \neq j$ in A-**smod**.

For M_i, there is no even isomorphism between M_i and ΠM_i, since the endomorphism ring $\mathrm{End}_A(M_i)$ is 1-dimensional and there should be no isomorphism of degree 1 in it. For N_j, we have an isomorphism between N_j and ΠN_j with degree 0, since there is an odd isomorphism in $\mathrm{End}_A(N_j)$.

Let S be an irreducible object in A-**emod**. Then S is isomorphic to M_i or N_j for some i or j in A-**smod**, and then is isomorphic to one from the set $\{M_i, \Pi M_i, N_j \mid i = 1, \cdots, l; j = 1, \cdots, q\}$ in A-**emod**. If S is isomorphic to M_i of degree 1 in A-**emod**, then S is isomorphic to ΠM_i in A-**emod**. For other three cases, we will get that S is isomorphic to M_i or N_j in A-**emod**. So $M_1, \cdots, M_l, \Pi M_1, \cdots, \Pi M_l, N_1, \cdots, N_q$ are all isomorphism classes of irreducible modules of A-**emod**. $\square$

The above lemma shows that the isomorphism classes of irreducible modules in A-**emod** can be recovered from those in A-**smod**.

Let σ be a k-linear involution of A defined by $\sigma(x) = (-1)^{\bar{x}} x$ for any homogeneous element $x \in A$ where $\bar{x} \in \mathbb{Z}_2$ denotes the parity of x. It is easy to see that σ induces an automorphism of order 2 of A. Let H be the group $\{1, \sigma\}$. We can form the skew group rings $\widetilde{A} := A\#H$, where $x\sigma = \sigma x$ and $y\sigma = -\sigma y$, for all $x \in A_{\bar{0}}$ and $y \in A_{\bar{1}}$. Then $\widetilde{A}$ becomes an ordinary associative k-algebra.

Proposition 7.2. *(cf.* [17, 2.12]*) The $\widetilde{A}$-module category $\widetilde{A}$-**Mod** is isomorphic to the even category A-**emod** of A-supermodules.*

7.2. **The skew group rings associated to $\mathfrak{g}$.** Take $\mathfrak{g} = \mathfrak{osp}(1|2n)$. We introduce an additional structure to $U(\mathfrak{g})$ for our further arguments. Recall $\mathfrak{g}$ is a restricted Lie superalgebra, with p-mapping $[p]$ on $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2n)$. For $\chi \in {\mathfrak{g}_{\bar{0}}}^*$, let $\mathfrak{m}_\chi$ be the ideal of $\mathcal{Z}_0$ generated by $x^p - x^{[p]} - \chi(x)^p$ for $x \in \mathfrak{g}_{\bar{0}}$. Then $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/\mathfrak{m}_\chi U(\mathfrak{g})$. Recall that $U(\mathfrak{g})$ has no zero-divsor (cf. Lemma 6.1).

We can form the skew group rings $\widetilde{U(\mathfrak{g})} := U(\mathfrak{g})\#H$ and $\widetilde{U_\chi(\mathfrak{g})} := U_\chi(\mathfrak{g})\#H$. In both of these rings, $x\sigma = \sigma x$ and $y\sigma = -\sigma y$, for all $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$.

Let $\widetilde{\mathcal{Z}}$ be the center of the algebra $U(\mathfrak{g})\#H$, and $\mathcal{A}(g)$ the anticenter of $U(g)$. First we have the relation between $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$ as follows.

Lemma 7.3. $\widetilde{\mathcal{Z}} = \mathcal{Z} + \sigma\mathcal{A}(g)$.

Proof. Let $z = x + \sigma y = x_0 + x_1 + \sigma(y_0 + y_1) \in \widetilde{\mathcal{Z}}$, where x_i, y_i are homogeneous elements of degree i, then

$$(x + \sigma y)(x' + \sigma y') = (x' + \sigma y')(x + \sigma y), \ \forall\, x', y' \in U(\mathfrak{g})$$

from which we have

$$\begin{cases} xx' + (y_0 - y_1)y' = x'x + (y'_0 - y'_1)y; \\ (x_0 - x_1)y' + yx' = y'x + (x'_0 - x'_1)y. \end{cases}$$

If $y' = 0$, then $x \in \mathcal{Z}$, so $x_1 = 0$; and then we have

$$\begin{cases} (y_0 - y_1)y' = (y'_0 - y'_1)y; \\ \qquad yx' = (x'_0 - x'_1)y. \end{cases}$$

If $x' = y'$ and $y'_0 = 0$, then $x'_1 y_1 = 0$ for any $x'_1 \in U(\mathfrak{g})_{\bar{1}}$, and hence $y_1 = 0$ since $U(\mathfrak{g})$ is a domain. If $x'_1 = 0$, then $yx'_0 = x'_0 y$, that is to say y commutes with the elements of $U(\mathfrak{g})_{\bar{0}}$. Now using the first equality above, we get $yy'_1 = -y'_1 y$ which means y anticommutes with the elements of $U(\mathfrak{g})_{\bar{1}}$, hence $y \in \mathcal{A}(\mathfrak{g})$, therefore $\widetilde{\mathcal{Z}} \subseteq \mathcal{Z} + \sigma\mathcal{A}(g)$. The inverse direction is obvious. □

Lemma 7.4. *The following statements hold:*

(1) $\widetilde{U(\mathfrak{g})}$ *and* $\widetilde{\mathcal{Z}}$ *are both Noetherian affine k-algebras. Moreover,* $\widetilde{U(\mathfrak{g})}$ *is finitely generated over* $\widetilde{\mathcal{Z}}$ *and* $\widetilde{\mathcal{Z}}$ *is integral over* $\mathcal{Z}_0$.

(2) $\widetilde{U(\mathfrak{g})}$ *is a prime ring.*

Proof. (1) Since $\mathcal{Z}_0 \subset \mathcal{Z} \subset \widetilde{\mathcal{Z}}$ and $\widetilde{U(\mathfrak{g})}$ is a free $\mathcal{Z}_0$-module with finite rank.

(2) Let $a = a_1 + a_2\sigma,\ b = b_1 + b_2\sigma \in \widetilde{U(\mathfrak{g})}$ with $acb = 0$ for any $c = c_1 + c_2\sigma \in \widetilde{U(\mathfrak{g})}$, we then have

$$\begin{cases} (a_1c_1 + a_2(c_2^0 - c_2^1))b_1 + (a_1c_2 + a_2(c_1^0 - c_1^1))(b_2^0 - b_2^1) = 0, \\ (a_1c_1 + a_2(c_2^0 - c_2^1))b_2 + (a_1c_2 + a_2(c_1^0 - c_1^1))(b_1^0 - b_1^1) = 0, \end{cases}$$

where $c_1^0,\ c_1^1$ are the homogeneous components of c_1.

In particular, if $c_1 = c_2 = 1$, then

$$\begin{cases} (a_1 + a_2)(b_1 + b_2^0 - b_2^1) = 0, \\ (a_1 + a_2)(b_2 + b_1^0 - b_1^1) = 0. \end{cases}$$

Since $U(\mathfrak{g})$ is a domain, we get $a_1 + a_2 = 0$, or $b_1^0 = -b_2^0$ *and* $b_1^1 = b_2^1$.

If $c_1 = 1,\ c_2 = -1$, then

$$\begin{cases} (a_1 - a_2)(b_1 - b_2^0 + b_2^1) = 0, \\ (a_1 - a_2)(b_2 - b_1^0 + b_1^1) = 0. \end{cases}$$

and we get $a_1 - a_2 = 0$, or $b_1^0 = b_2^0$ and $b_1^1 = -b_2^1$.

If $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$, then $a_1 = a_2 = 0$, so $a = 0$;

If $a_1+a_2=0$ and $b_1^0=b_2^0$ and $b_1^1=-b_2^1$, then $b_2\sigma=\sigma b_1$. In this case $a_1(1-\sigma)(c_1+c_2\sigma)(1+\sigma)b_1=0$, for any $c_1,c_2\in U(\mathfrak{g})$. If $c_2=0$ and $0\neq c_1\in U(\mathfrak{g})_{\bar{1}}$, we have

$$2a_1c_1b_1+2a_1c_1\sigma b_1=0,$$

from which we get $a_1=0$, or $c_1b_1=0$. Since $c_1\neq 0$, we have $a_1=0$ or $b_1=0$. Hence $a=a_1(1-\sigma)=0$ or $b=(1+\sigma)b_1=0$. Similarly, we can also get $a=0$ or $b=0$ for the other two cases. Therefore, $\widetilde{U(\mathfrak{g})}$ is prime. □

In order to make use of the algebra $\widetilde{U(\mathfrak{g})}$, we need to know more about the relations between the representation categories of $U(\mathfrak{g})$ and of $\widetilde{U(\mathfrak{g})}$.

As used before, denote by $\widetilde{U_\chi(\mathfrak{g})}$-**Mod** the category of $\widetilde{U_\chi(\mathfrak{g})}$-modules. For any $M\in \mathrm{Ob}(\widetilde{U_\chi(\mathfrak{g})}\text{-}\mathbf{Mod})$, M can be decomposed into the sum of eigenvalue spaces of σ. Since $\sigma^2=\mathrm{id}$, the eigenvalues are 1 and -1. Let M_i denote the eigenvalue space corresponding to eigenvalue i , $i=1,-1$. Define a correspondence F : $\widetilde{U_\chi(\mathfrak{g})}$-**Mod** $\longrightarrow U_\chi(\mathfrak{g})$-**emod** via $F(M)_{\bar{0}}=M_1$ and $F(M)_{\bar{1}}=M_{-1}$. And define $G: U_\chi(\mathfrak{g})$-**emod**$\longrightarrow \widetilde{U_\chi(\mathfrak{g})}$-**Mod** via $\sigma|_{M_{\bar{0}}}=\mathrm{id}$ and $\sigma|_{M_{\bar{1}}}=-\mathrm{id}$. Obviously, F and G are functors of categories. Proposition 7.2 says that the functors F and G are quasi-inverse.The category $\widetilde{U_\chi(\mathfrak{g})}$-**Mod** is isomorphic to the underlying even category of $U_\chi(\mathfrak{g})$-**emod**. Furthermore, the following lemma is clear.

Lemma 7.5. *For any $\chi\in\mathfrak{g}_{\bar{0}}{}^*$, there is an isomorphism between $\widetilde{U_\chi(\mathfrak{g})}$ and the quotient algebra $\widetilde{U(\mathfrak{g})}/\mathfrak{m}_\chi\widetilde{U(\mathfrak{g})}$, where $\mathfrak{m}_\chi$ is the ideal of $\mathcal{Z}_0$ generated by $x^p-x^{[p]}-\chi(x)^p$ for $x\in\mathfrak{g}_{\bar{0}}$.*

Proof. There is a natural map

$$\varphi:\ \widetilde{U_\chi(\mathfrak{g})}\longrightarrow \widetilde{U(\mathfrak{g})}/\mathfrak{m}_\chi\widetilde{U(\mathfrak{g})}$$

given by

$$\varphi(\bar{a}+\bar{b}\sigma)=\overline{a+b\sigma},\ \forall\ a,b\in U(\mathfrak{g}).$$

It is easy to show that this map is well-defined and it is indeed an isomorphism of algebras. □

7.3. **Irreducible modules for $\mathfrak{g}=\mathfrak{osp}(1|2)$.** From now on, we turn to the case when $\mathfrak{g}=\mathfrak{osp}(1|2)$. Recall that $\mathfrak{g}=\mathfrak{osp}(1|2)$ consists of 3×3 matrices in the following $(1|2)$-block form

$$\begin{pmatrix}0&v&u\\u&a&b\\-v&c&-a\end{pmatrix}$$

with $a,b,c,u,v\in k$. The even part is spanned by

$$e=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad h=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad f=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}.$$

A basis of the odd part is given by

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We collect the commutation relations of these basis elements as below:

$$\begin{gathered} [h,E]=E,\ [e,E]=0,\ [f,E]=-F; \\ [h,F]=-F,\ [e,F]=-E,\ [f,F]=0; \\ [E,E]=2e,\ [E,F]=h,\ [F,F]=-2f \end{gathered}$$

By the above relations we can easily show that $S = EF - FE + \frac{1}{2}$ commutes with the even part, and anticommutes with the odd part. So, $S\sigma \in \mathcal{Z}$.

Since $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{sl}(2)$, there are three coadjoint orbits of $\mathfrak{g}_{\bar{0}}{}^*$ with the following representatives:

(i) regular nilpotent : $\chi_0(e) = \chi_0(h) = 0$ and $\chi_0(f) = 1$;
(ii) regualr semisimple : $\chi_1(e) = \chi_1(f) = 0$ and $\chi_1(h) = a^p$ for some $a \in k^\times$;
(iii) restricted : $\chi_2(e) = \chi_2(h) = \chi_2(f) = 0$, *i.e.* $\chi_2 = 0$

Recall that for χ from the above (i)-(iii), one has a baby Verma module $Z_\chi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} k_\lambda$ where $\mathfrak{b} := ke + kh + kE$ with λ satisfying $\lambda^p - \lambda = \chi(h)^p$. Then $Z_\chi(\lambda)$ has a basis $v_i := F^i \otimes 1, i = 0, 1, \cdots, 2p-1$ with $\mathfrak{g}$-action as listed as in [38, §6.2]. For the readers' convenience, we list the descriptions of isomorphism classes of irreducible modules for $\mathfrak{osp}(1|2)$ as below (modulo equivalence arising from the G_{ev}-conjugation).

Proposition 7.6. *(cf.* [38, §6]*) Let* $\mathfrak{g} = \mathfrak{osp}(1|2)$*. The following statements hold.*

1. *The superalgebra* $U_{\chi_0}(\mathfrak{g})$ *has* $\frac{p+1}{2}$ *distinct isomorphism classes of irreducible modules, which are,* $Z_{\chi_0}(\lambda)$ *for* $\lambda = 0, 1, \cdots, \frac{p-1}{2}$*, with* $Z_{\chi_0}(\lambda) \cong Z_{\chi_0}(p - \lambda - 1)$*. All those irreducible modules are of type* M *except* $\lambda = \frac{p-1}{2}$*. The exceptional one is of type* Q*.*
2. *The superalgebra* $U_{\chi_1}(\mathfrak{g})$ *has* p *distinct isomorphism classes of irreducible modules, represented by* $Z_{\chi_1}(\lambda)$ *with* $\lambda^p - \lambda = a^p$*. All those irreducible modules are of type* M*.*
3. *The superalgebra* $U_0(\mathfrak{g})$ *has* p *distinct isomorphism classes of irreducible modules* $L(\lambda)$ *for* $\lambda = 0, \cdots, p-1$*, which is the unique irreducible quotient of* $Z_0(\lambda)$*. Moreover,* $L(\lambda)$ *has dimension* $2\lambda + 1$*.*

7.4. Continue to work with $\mathfrak{g} = \mathfrak{osp}(1|2)$ and keep the notations as in the above subsections. Set $U = G_{\mathrm{ev}} \cdot \chi_1$, and $V = G_{\mathrm{ev}} \cdot \chi_2$. Note that $U \cup V$ is the set of all regular elements of $\mathfrak{g}_{\bar{0}}^*$ and $\dim(U \cup V) = \dim \mathfrak{g}_{\bar{0}}^* = 3$.

Recall that a ring R with center C is called an Azumaya algebra over C if and only if R is a finitely generated projective C-module and the natural map $R \otimes R^{\mathrm{op}} \longrightarrow \mathrm{End}_C(R)$ is an isomorphism. Define

$$\mathsf{A} := \{\mathfrak{m} \in \mathrm{Maxspec}(\widetilde{\mathcal{Z}}) \mid \widetilde{U(\mathfrak{g})}_{\mathfrak{m}} \text{ is Azumaya over } \widetilde{\mathcal{Z}}_{\mathfrak{m}}\}, \tag{7.1}$$

as in [7, §3.2]. The set A is called the Azumaya locus of $\widetilde{U(\mathfrak{g})}$.

By Lemma 7.4, $\widetilde{U(\mathfrak{g})}$ is a prime Noetherian ring which is module-finite over its center $\widetilde{\mathfrak{Z}}$. Thus, Proposition 3.1 in [7] tells us that $\widetilde{U(\mathfrak{g})}_{\mathfrak{m}}$ is an Azumaya algebra over $\widetilde{\mathfrak{Z}}_{\mathfrak{m}}$ if and only $\mathfrak{m} = \widetilde{\mathfrak{Z}} \cap \operatorname{Ann} M$ for some irreducible module M of maximal dimension.

Note that each irreducible $\widetilde{U(\mathfrak{g})}$-module becomes in a natural way an irreducible $\widetilde{U_\chi(\mathfrak{g})}$-module for some $\chi \in \mathfrak{g}_{\bar{0}}{}^*$. By Lemmas 7.1 and 7.5, the maximal dimension of irreducible $\widetilde{U(\mathfrak{g})}$-modules over k equals to the maximal dimension of irreducible $U(\mathfrak{g})$-modules over k. And this dimension is $2p$ (cf. Proposition 7.6). So the dimension of all irreducible $\widetilde{U_\chi(\mathfrak{g})}$-modules are $2p$ for $\chi \in U \cup V$ and the dimension of all irreducible $\widetilde{U_0(\mathfrak{g})} := U_0(\mathfrak{g})\#H$-modules are less than $2p$ for $\chi = 0$. We have the following lemma.

Lemma 7.7. *Keep the notations as before. The following statements hold:*

(1) $\mathsf{A} = \{\mathfrak{m} \in \mathit{Maxspec}(\widetilde{\mathfrak{Z}}) \mid \mathfrak{m} \cap \mathfrak{Z}_0 = \mathfrak{m}_\chi, \forall\ \chi \in U \cup V\}$*, where m_χ is the same as in Lemma 7.5.*

(2) *The locus of non-Azumaya points of $\widetilde{\mathfrak{Z}}$ has codimension 3 in Maxspec($\widetilde{\mathfrak{Z}}$).*

Proof. (1) For any given $\chi \in \mathfrak{g}_{\bar{0}}{}^*$, there is a k-algebra morphism $\xi : \widetilde{\mathfrak{Z}} \longrightarrow k$ such that $\xi|_{\mathfrak{Z}_0} = \chi^p$. Then $\mathfrak{m} := \ker \xi \in \operatorname{Maxpsec}(\widetilde{\mathfrak{Z}})$, and $\mathfrak{m} \cap \mathfrak{Z}_0 = \mathfrak{m}_\chi$. Let k_ξ stand for the corresponding one-dimensional $\widetilde{\mathfrak{Z}}$-module. Consider $\widetilde{Z_\xi} := \widetilde{U(\mathfrak{g})} \otimes_{\widetilde{\mathfrak{Z}}} k_\xi$. Note that $\widetilde{U(\mathfrak{g})}$ is module-finite over $\widetilde{Z}$. So $\widetilde{Z_\xi}$ admits an irreducible quotient $\widetilde{U(\mathfrak{g})}$-module, say M. Then M is actually an irreducible $\widetilde{U_\chi(\mathfrak{g})}$-module. Obviously, $\mathfrak{m} = \operatorname{Ann}_{\widetilde{U(\mathfrak{g})}}(M) \cap \widetilde{\mathfrak{Z}}$.

By Proposition 7.6 and the arguments before the lemma, M has maximal dimension if and only if $\chi \in U \cup V$. Statement (1) follows.

(2) Set $X := \{\mathfrak{m}_\chi \mid \chi \in U \cup V\}$. Then X is an open subset of $\operatorname{Maxspec}(\mathfrak{Z}_0)$ with complement of codimension 3. Since $\widetilde{\mathfrak{Z}}$ is module-finite over $\mathfrak{Z}_0$, the restriction map $\gamma : \operatorname{Maxspec}(\widetilde{\mathfrak{Z}}) \longrightarrow \operatorname{Maxspec}(\mathfrak{Z}_0)$ has finite fibres; therefore $\gamma^{-1}(X)$ is an open subset of $\operatorname{Maxspec}(\widetilde{\mathfrak{Z}})$ with complement of codimension 3. By (1), $\gamma^{-1}(X)$ is exactly the Azumaya locus of $\widetilde{U(\mathfrak{g})}$, and (2) follows. $\square$

Proposition 7.8. *Let $\mathfrak{m} \in Maxspec(\widetilde{\mathfrak{Z}})$. Then $\widetilde{U(\mathfrak{g})}_{\mathfrak{m}}$ is projective over $\widetilde{\mathfrak{Z}}_{\mathfrak{m}}$ if and only if $\widetilde{U(\mathfrak{g})}_{\mathfrak{m}}$ is Azumaya over $\widetilde{\mathfrak{Z}}_{\mathfrak{m}}$, i.e. $\mathfrak{m} \in \mathsf{A}$.*

Proof. By the arguments in the proof of Lemma 7.7, we can assume that $\mathfrak{m} = \widetilde{\mathfrak{Z}} \cap \operatorname{Ann}_{\widetilde{U(\mathfrak{g})}}(M)$ for some irreducible $\widetilde{U(\mathfrak{g})}$-module M. Assume that $\widetilde{U(\mathfrak{g})}_{\mathfrak{m}}$ is projective over $\widetilde{\mathfrak{Z}}_{\mathfrak{m}}$. By [7, Lemma 3.6] it is enough to show the non-Azumaya locus of $\widetilde{\mathfrak{Z}}$ has codimension at least 2. This is ensured by Lemma 7.7(2). Conversely, from the definition of Azumaya algebra it follows that $\widetilde{U(\mathfrak{g})}_{\mathfrak{m}}$ is projective over $\widetilde{\mathfrak{Z}}_{\mathfrak{m}}$ for $\mathfrak{m} \in \mathsf{A}$. $\square$

7.5. Recall that $S = EF - FE + \frac{1}{2} \in A(g)$. And then $S\sigma \in Z$. Applying Corollary 6.6 to $\mathfrak{g} = \mathfrak{osp}(1|2)$, one easily has the following observation.

Lemma 7.9. *Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. Then $\mathcal{Z}$ is generated by $\mathcal{Z}_0$ and S^2.*

Lemma 7.10. *Maintain the notations as above, then $\mathcal{A}(\mathfrak{g}) = \mathcal{Z}S$.*

Proof. Assume that $T \in \mathcal{A}(\mathfrak{g})$, then $TS \in \mathcal{Z}$, i.e. $TS = a_n S^{2n} + \cdots + a_1 S^2 + a_0$ where $a_0, \cdots, a_n \in \mathcal{Z}_0$ for some $n \in \mathbb{N}$. So $(T - a_n S^{2n-1} - \cdots - a_1 S)S = a_0$.

If $a_0 = 0$, then $T = a_n S^{2n-1} - \cdots - a_i S^{2i-1} - \cdots - a_1 S \in \mathcal{Z}S$, which is desired.

If $a_0 \neq 0$, we replace T by $T - a_n S^{2n-1} - \cdots - a_1 S$. The new T is still in $\mathcal{A}(\mathfrak{g})$, satisfying $TS = a_0 \neq 0$. We only need to show that the new T is in $\mathcal{Z}S$. Since $\mathcal{A}(\mathfrak{g}) \in U(\mathfrak{g})_{\bar{0}}$, we can assume $T = u_0 + u_3 EF$ where $u_0, u_3 \in U(\mathfrak{g}_{\bar{0}})$. Then

$$TS = (-u_0 h + \frac{1}{2}u_0 + 2u_3 ef) + (2u_0 + u_3(h-2) + \frac{1}{2}u_3)EF = a_0,$$

which implies that

$$-u_0 h + \frac{1}{2}u_0 + 2u_3 ef = a_0, (2u_0 + u_3(h-2) + \frac{1}{2}u_3)EF = 0.$$

So $2u_0 + u_3(h-2) + \frac{1}{2}u_3 = 0$ i.e. $2u_0 = -u_3(h - \frac{3}{2})$, then

$$-u_0 h + \frac{1}{2}u_0 + 2u_3 ef = \frac{u_3}{2}(h^2 + 2h + 4fe + \frac{3}{4}) = \frac{u_3}{2}(S^2 + S).$$

Denote $\frac{1}{8}((E_{11} - E_{22})^2 + 2(E_{11} - E_{22}) + 4E_{21}E_{12})$ by Ω. Note that Ω is a Casimir element of the Lie algebra $\mathfrak{sl}(2)$, and the center of $U(\mathfrak{sl}(2))$ is generated by $\mathcal{Z}_0$ and Ω. In our case $\Omega = \frac{1}{8}(h^2 + 2h + 4fe)$, so $S^2 + S = 8\Omega + \frac{3}{4}$. Hence we can replace Ω by $S^2 + S$. Thus

$$TS = \frac{u_3}{2}(S+1)S = a_0 \in \mathcal{Z}_0 \text{ therefore } u_3 \in \mathcal{Z}(U(\mathfrak{g}_{\bar{0}})).$$

We write

$$\frac{u_3}{2} = b_m \Omega^m + \cdots + b_1 \Omega + b_0 = b_m (S^2 + S)^m + \cdots + b_1 (S^2 + S) + b_0 = x + yS$$

where $b_i \in \mathcal{Z}_0$. Then $x, y \in \mathcal{Z}$, therefore

$$T = (x + yS)(S + 1) = (x + y)S + (x + yS^2).$$

Since $x + yS^2 \in \mathcal{Z}$, we have $T = (x + y)S \in \mathcal{Z}S$. □

Corollary 7.11. *$\widetilde{\mathcal{Z}}$ is a free $\mathcal{Z}$-module.*

Proof. This follows immediately from Lemmas 7.3 and 7.10. □

7.6. Let's recall some local properties of Noetherian rings.

Lemma 7.12. *(cf.* [2, Proposition 3.10]*) Let R be a Noetherian commutative ring, M a finitely generated R-module. Then the following conditions are equivalent*

(i) M is projective;

(ii) M is flat;

(ii) $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p}$;

(iii) $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$.

Proposition 7.13. *Let M be an irreducible $U_\chi(\mathfrak{g})$-module with $\chi\in U\cup V$. Then $U(\mathfrak{g})_\mathfrak{m}$ is projective over $\mathcal{Z}_\mathfrak{m}$ for $\mathfrak{m}=\mathrm{Ann}_\mathcal{Z}(M)$,*

Proof. Note that the dimension of irreducible $\widetilde{U(\mathfrak{g})}$-module M is maximal. So $\widetilde{U(\mathfrak{g})}_{\widetilde{\mathfrak{m}}}$ is Azumaya over $\widetilde{\mathcal{Z}}_{\widetilde{\mathfrak{m}}}$ for $\widetilde{\mathfrak{m}}=\mathrm{Ann}_{\widetilde{\mathcal{Z}}}(M)$. By Proposition 7.8,

$$\widetilde{U(\mathfrak{g})}_{\widetilde{\mathfrak{m}}} \text{ is finite free } \widetilde{\mathcal{Z}}_{\widetilde{\mathfrak{m}}}\text{-module.} \tag{7.2}$$

The same thing as (7.2) happens with respect to any irreducible $U_\chi(\mathfrak{g})$-module N for $\chi\in U\cup V$. Note that the maximal ideals of $\widetilde{\mathcal{Z}}_\mathfrak{m}$ are of the form $Q_\mathfrak{m}$ with $Q=\mathrm{Ann}_{\widetilde{\mathcal{Z}}}(N)$ for some irreducible $U_\chi(\mathfrak{g})$-module, $\chi\in U\cup V$ (cf. Lemma 7.7). And then $\widetilde{U(\mathfrak{g})}_Q$ is finite free $\widetilde{\mathcal{Z}}_Q$-module for each maximal ideal Q of $\widetilde{\mathcal{Z}}$ (cf. (7.2)). From Lemma 7.12 it follows that $\widetilde{U(\mathfrak{g})}_\mathfrak{m}$ is projective over $\widetilde{\mathcal{Z}}_\mathfrak{m}$. By Corollary 7.11, $\widetilde{U(\mathfrak{g})}_\mathfrak{m}$ is projective over $\mathcal{Z}_\mathfrak{m}$. Now $U(\mathfrak{g})_\mathfrak{m}$ is a direct summand of $\widetilde{U(\mathfrak{g})}_\mathfrak{m}$ as a $\mathcal{Z}_\mathfrak{m}$-module. Hence, $U(\mathfrak{g})_\mathfrak{m}$ is projective over $\mathcal{Z}_\mathfrak{m}$. □

As listed in Proposition 7.6, the p isomorphism classes of irreducible $U_0(\mathfrak{g})$-modules are exactly $L(\lambda),\ \lambda=0,\cdots,p-1$ with $\dim L(\lambda)=2\lambda+1<2p$.

Proposition 7.14. *$U(\mathfrak{g})_\mathfrak{m}$ is not projective over $\mathcal{Z}_\mathfrak{m}$ for those maximal ideals $\mathfrak{m}=\mathrm{Ann}_\mathcal{Z}(L(\lambda))$, where $\lambda\neq\frac{p-1}{2}$.*

Proof. Let $\mathrm{V}=\{L(0),\cdots,L(p-1),\Pi L(0),\cdots,\Pi L(p-1)\}$ and $P=\mathrm{Ann}_{\widetilde{\mathcal{Z}}}(M)$ for $M\in\mathrm{V}$. All those M fail to be irreducible modules of maximal dimension. Set $\mathfrak{m}=P\cap\mathcal{Z}$. Note that we have shown $\widetilde{U(\mathfrak{g})}_P$ is not Azumaya over $\widetilde{\mathcal{Z}}_P$ (cf. [7, Proposition 3.1]). By Lemma 7.12 and Proposition 7.8, it is enough to show that $\widetilde{U(\mathfrak{g})}_\mathfrak{m}$ is a free $\mathcal{Z}_\mathfrak{m}$ module if $U(\mathfrak{g})_\mathfrak{m}$ is projective over $\mathcal{Z}_\mathfrak{m}$.

Suppose $U(\mathfrak{g})_\mathfrak{m}$ is finitely generated projective over $\mathcal{Z}_\mathfrak{m}$. Since $\mathcal{Z}_\mathfrak{m}$ is local, $U(\mathfrak{g})_\mathfrak{m}$ is a finite free $\mathcal{Z}_\mathfrak{m}$-module. Let $u_1,\cdots,u_r$ be a basis of $U(\mathfrak{g})_\mathfrak{m}$ over $\mathcal{Z}_\mathfrak{m}$. Note that S^2 acts on $L(\lambda)$ by the scalar $(\lambda+\frac{1}{2})^2$, hence $S^2\notin\mathfrak{m}$. Thus $\frac{1}{S^2}\in\mathcal{Z}_\mathfrak{m}$. Assume that $S\big(u_1,u_2,\cdots,u_r\big)=\big(u_1,u_2,\cdots,u_r\big)D$ where D is a $(r\times r)$-matrix with coefficients in $\mathcal{Z}_\mathfrak{m}$, so $S^2\big(u_1,u_2,\cdots,u_r\big)=\big(u_1,u_2,\cdots,u_r\big)D^2$, which implies $D^2=S^2I$ where I is the $(r\times r)$-identity matrix since $S^2\in\mathcal{Z}$.

Let x be any element in $U(\mathfrak{g})_\mathfrak{m}$. Thus there must exist $c_1,\cdots,c_r\in\mathcal{Z}_\mathfrak{m}$ such that $x=\sum\limits_{i=1}^r c_iu_i$, and then

$$\begin{aligned} x &= \big(u_1,u_2,\cdots,u_r\big)\begin{pmatrix}c_1\\c_2\\\vdots\\c_r\end{pmatrix}=\big(u_1,u_2,\cdots,u_r\big)\frac{1}{S^2}D^2\begin{pmatrix}c_1\\c_2\\\vdots\\c_r\end{pmatrix}\\ &= \big(Su_1,Su_2,\cdots,Su_r\big)\frac{1}{S^2}D\begin{pmatrix}c_1\\c_2\\\vdots\\c_r\end{pmatrix}=\sum_{i=1}^r d_iSu_i=S\sum_{i=1}^r d_iu_i,\end{aligned}$$

where $(d_1, d_2, \cdots, d_r) = \frac{1}{S^2}(c_1, c_2, \cdots, c_r)D^{\mathrm{t}}$, and therefore $d_i \in \mathcal{Z}_{\mathfrak{m}}$.

Thus for any $x + \sigma x' \in \widetilde{U(\mathfrak{g})_{\mathfrak{m}}}$ with $x, x' \in U(\mathfrak{g})_{\mathfrak{m}}$, there exist $c_i, c_i' \in \mathcal{Z}_{\mathfrak{m}},\ i = 1, \cdots, r$ such that $x = \sum\limits_{i=1}^{r} c_i u_i$ and $x' = \sum\limits_{i=1}^{r} Sc_i' u_i$. Hence

$$x + \sigma x' = \sum_{i=1}^{r} c_i u_i + \sigma \sum_{i=1}^{r} Sc_i' u_i = \sum_{i=1}^{r} (c_i + \sigma Sc_i') u_i,$$

which implies that $u_1, \cdots, u_r$ generate $\widetilde{U(\mathfrak{g})_{\mathfrak{m}}}$ over $\widetilde{\mathcal{Z}}_{\mathfrak{m}}$. We want to show further that $\{u_1, \cdots, u_r\}$ is a basis of $\widetilde{U(\mathfrak{g})_{\mathfrak{m}}}$ over $\widetilde{\mathcal{Z}}_{\mathfrak{m}}$. If $\sum\limits_{i=1}^{r}(c_i + \sigma S d_i) u_i = 0$ for some $c_i, d_i \in \mathcal{Z}_{\mathfrak{m}}$, then $\sum\limits_{i=1}^{r} c_i u_i = 0$ and $\sum\limits_{i=1}^{r} d_i u_i = 0$. By the freeness of $u_1, \cdots, u_r$ over $\mathcal{Z}_{\mathfrak{m}}$, we get $c_i = d_i = 0$ for any i. So $\{u_1, \cdots, u_r\}$ is indeed a basis of $\widetilde{U(\mathfrak{g})_{\mathfrak{m}}}$ over $\widetilde{\mathcal{Z}}_{\mathfrak{m}}$. Thus the proposition is proved, in view of the analysis in the beginning. $\square$

7.7. We are in the position to state our main theorem of this section. Let $\mathsf{B} = \{\mathrm{Ann}_{\mathcal{Z}}(M) \mid M \text{ is a simple } U_\chi(\mathfrak{g}) \text{ module with } \chi \text{ regular}\}$.

Theorem 7.15. *The smooth locus of $\mathcal{Z}(\mathfrak{g})$ are the union of B and* $\mathrm{Ann}_{\mathcal{Z}}(L(\frac{p-1}{2}))$.

Proof. We first claim that $\mathcal{Z}_{\mathfrak{m}}$ is regular if and only if $U(\mathfrak{g})_{\mathfrak{m}}$ is projective over $\mathcal{Z}_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Maxspec}(\mathcal{Z})$. By Lemma 6.2, the "only if" part follows from [7, Proposition 3.7]. For the other part, we assume that $U(\mathfrak{g})_{\mathfrak{m}}$ is projective $\mathcal{Z}_{\mathfrak{m}}$-module. Then $U(\mathfrak{g})_{\mathfrak{m}}$ is finite free over $\mathcal{Z}_{\mathfrak{m}}$ since $\mathcal{Z}_{\mathfrak{m}}$ is local. Write $U(\mathfrak{g})_{\mathfrak{m}} = \bigoplus^r \mathcal{Z}_{\mathfrak{m}}$. Then $U(\mathfrak{g})_{\mathfrak{m}}/\mathfrak{m}U(\mathfrak{g})_{\mathfrak{m}} \cong \bigoplus^r \mathcal{Z}_{\mathfrak{m}}/\mathfrak{m}\mathcal{Z}_{\mathfrak{m}}$. A finite projective $U(\mathfrak{g})_{\mathfrak{m}}$-resolution of $U(\mathfrak{g})_{\mathfrak{m}}/\mathfrak{m}U(\mathfrak{g})_{\mathfrak{m}}$ affords a finite free $\mathcal{Z}_{\mathfrak{m}}$-resolution of $U(\mathfrak{g})_{\mathfrak{m}}/\mathfrak{m}U(\mathfrak{g})_{\mathfrak{m}}$. Since $\mathcal{Z}_{\mathfrak{m}}/\mathfrak{m}\mathcal{Z}_{\mathfrak{m}}$ is a direct summand of $U(\mathfrak{g})_{\mathfrak{m}}/\mathfrak{m}U(\mathfrak{g})_{\mathfrak{m}}$, it follows that $\mathcal{Z}_{\mathfrak{m}}/\mathfrak{m}\mathcal{Z}_{\mathfrak{m}}$ has finite projective dimension. Thus $\mathcal{Z}_{\mathfrak{m}}$ is regular by Serre Theorem ([28, Theorem 9.58]).

By Corollary 6.6, $\mathcal{Z}(\mathfrak{g})$ is isomorphic to $\mathcal{Z}(\mathfrak{g}_{\bar{0}}) = \mathcal{Z}(\mathfrak{sl}(2))$. As we know, there are exactly $p-1$ singular points in $\mathcal{Z}(\mathfrak{sl}(2))$ (cf. [19, §5] and [6, §3.11 and §3.15]). Now by Proposition 7.14, there are already $p-1$ singular points of $\mathcal{Z}(\mathfrak{g})$. So the theorem follows. $\square$

Remark 7.16. From the above theorem, we know that the relation between the Azumaya locus and the smooth locus of $\mathrm{Maxspec}(\mathcal{Z})$ for classical Lie superalgebra is different from the situation for classical Lie algebras. In addition, Proposition 7.13 gives us another way to judge the smoothness of those points corresponding to irreducible modules with maximal dimension.

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